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String Windings in the Early Universe

Richard Easther[a], Brian R. Greene[bc], Mark G. Jackson[bd], Daniel Kabat[be]

[a]*Department of Physics*
Yale University, New Haven, CT 06520
`richard.easther@yale.edu`

[b]*Institute for Strings, Cosmology and Astroparticle Physics*
and Department of Physics
Columbia University, New York, NY 10027
`greene, markj, kabat@physics.columbia.edu`

[c]*Department of Mathematics*
Columbia University, New York, NY 10027

[d]*NASA/Fermilab Theoretical Astrophysics Group*
Fermilab, Batavia, IL 60510

[e]*Department of Physics and Astronomy*
Rutgers University, Piscataway, NJ 08855-0849

We study string dynamics in the early universe. Our motivation is the proposal of Brandenberger and Vafa, that string winding modes may play a key role in decompactifying three spatial dimensions. We model the universe as a homogeneous but anisotropic 9-torus filled with a gas of excited strings. We adopt initial conditions which fix the dilaton and the volume of the torus, but otherwise assume all states are equally likely. We study the evolution of the system both analytically and numerically to determine the late-time behavior. We find that, although dynamical evolution can indeed lead to three large spatial dimensions, such an outcome is not statistically favored.

1 Introduction

An enduring challenge for string/M-theory is to provide a more complete picture of the early universe than has been found using conventional, point-particle approaches. To this end, a growing body of research has studied the dynamics of strings and branes in a cosmological setting, as opposed to the more widely investigated case of a static background. Intriguing but as yet incomplete results have been found for higher dimensional cosmologies and cosmologies based on braneworlds. These include mechanisms for resolving or avoiding cosmological singularities, and for generating subtle modifications to the primordial microwave background power spectrum. Further progress on these key theoretical and observational issues, however, requires a more refined grasp of the dynamical properties of strings and branes when subject to extremes of temperature, density, and curvature. The current paper provides a modest step in this direction.

The formalism we develop can, in principle, be applied to a wide range of string cosmology questions. But following our earlier works [1] [2], our immediate goal is to find a dynamical mechanism within string/M-theory that generically gives rise to a universe with precisely three large spatial dimensions, with all other spatial dimensions unobservably small. Such an asymmetric dynamical evolution is perhaps the most basic task of string/M cosmology. However, a decade and a half after the first attempt, no satisfactory picture has yet emerged.

By way of brief history, in [3] and [4] the authors made use of T-duality in a (spatially) toroidal universe to argue that strings wound around nontrivial cycles impede the growth rate of the spatial dimensions they wrap. The fastest expansion will therefore be achieved by dimensions that shed all their winding modes through string winding/anti-winding annihilations. Because string worldsheets are two-dimensional, pairs of strings will generically intersect in four or fewer spacetime dimensions, leading [3] and [4] to argue that at most three spatial dimensions will shed their windings and subsequently grow with time. Various aspects of this proposal have been investigated and generalized. In [5], a numerical study of a gas of strings in a toroidal universe was carried out and the naive dimension counting argument for string annihilations used in [3] (2+2 = 3+1) was verified in a static background. In [6], the analysis was further extended to simply connected toroidal orbifolds, and it was argued that pseudo-winding modes with sufficiently long lifetimes could allow the arguments of [3] to apply in phenomenologically relevant backgrounds. In [7], higher dimensional branes were included in the analysis, without invalidating the conclusions of [3]. Other studies of string windings and brane gases can be found in [8]-[24].

All these works, however, fail to account for the detailed cosmological dynamics. In [1] we partially addressed this deficiency by using eleven-dimensional supergravity to study cosmological evolution in the presence of a brane gas and found encouraging results: for a suitable configuration of brane wrappings suggested in [7] and [1] and based on naive dimension counting arguments, the dynamics does indeed drive an asymmetric evolution yielding three large dimensions. In [2] we went further and studied the coupled Einstein-Boltzmann equations for a thermal brane gas and found that despite the naive dimension counting arguments, only highly specialized initial conditions yield the desired brane wrapping configuration. In particular, we found that the spatial expansion driven by the brane gas is generically too fast for brane interactions to generate the expected anisotropies; instead, the branes quickly freeze out. However, this analysis still held out the possibility of a loop-hole that would allow one to evade this discouraging conclusion. In the string theory corner of M-theory moduli space (the very scenario studied in Brandenberger and Vafa's initial paper [3]), the growth of spatial dimensions appeared to be slower, perhaps allowing sufficient string-string interactions to yield the desired asymmetric winding configuration and hence asymmetric expansion. The main purpose of this paper is to study this possibility in detail.

In section 2 we set up the basic framework of dilaton gravity, and in section 3 we discuss the equilibrium thermodynamics of a string gas. In section 4 we introduce the Boltzmann equations which govern string annihilation and give a preliminary discussion of the phenomenon of freeze-out. In section 5 we describe our method for sampling from the possible initial states of the universe, and discuss holographic bounds on the space of initial conditions. In section 6 we present numerical results, and show that other than for a narrow range of initial conditions we get "all" or "nothing" evolution: either there are too few strings to keep any spatial dimensions small, or string interactions freeze out so quickly that a large number of wrapped strings survive and prevent any dimensions from growing large.

2 Dilaton gravity

We start with type II string theory compactified on a 9-torus, with metric

$$ds^2 = -dt^2 + \alpha' \sum_{i=1}^{9} e^{2\lambda_i(t)} d\theta_i^2 \qquad 0 \le \theta_i \le 2\pi\,. \tag{1}$$

The action for dilaton gravity is

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g}\, e^{-2\phi} \left(R + 4(\partial\phi)^2 + \cdots\right) \tag{2}$$

where (Polchinski [25], 13.3.24) $\kappa_{10}^2 = \frac{1}{2}(2\pi)^7(\alpha')^4$. From now on we set $\alpha' = 1$. Following Tseytlin & Vafa [4], we define the shifted dilaton

$$\varphi = 2\phi - \sum_i \lambda_i \tag{3}$$

so that the action reads

$$S = (2\pi)^2 \int dt\, e^{-\varphi} \Big(\sum_i \dot{\lambda}_i^2 - \dot{\varphi}^2\Big)\,. \tag{4}$$

When one couples dilaton gravity to a matter system the time-time component of the Einstein equations yields the Hamiltonian constraint (or Friedman equation)

$$(2\pi)^2 e^{-\varphi} \Big(\dot{\varphi}^2 - \sum_i \dot{\lambda}_i^2\Big) = E \tag{5}$$

where E is the total matter energy. This constraint implies that $\dot{\varphi}^2$ never vanishes; we choose the direction of time so that $\dot{\varphi} < 0$. The dilaton equation of motion is

$$\ddot{\varphi} = \frac{1}{2}\Big(\dot{\varphi}^2 + \sum_i \dot{\lambda}_i^2\Big)\,. \tag{6}$$

The scale factors obey

$$\ddot{\lambda}_i - \dot{\varphi}\dot{\lambda}_i = \frac{1}{8\pi^2} e^{\varphi} P_i \tag{7}$$

where the "total pressure" $P_i = -\frac{\partial F}{\partial \lambda_i}$ is obtained by varying the matter free energy with respect to λ_i. P_i is equal to the ordinary pressure in the i^{th} direction times the spatial volume.

Note that T-duality in the ith direction takes the simple form

$$\lambda_i \to -\lambda_i, \qquad \varphi \text{ invariant}.$$

This leaves the dynamics unchanged, provided that E is invariant and P_i changes sign.

To get oriented we first study the vacuum equations, with all pressures set to zero. The equations of motion reduce to

$$\ddot{\varphi} = \frac{1}{2}\Big(\dot{\varphi}^2 + \sum_i \dot{\lambda}_i^2\Big) \tag{8}$$

$$\ddot{\lambda}_i - \dot{\varphi}\dot{\lambda}_i = 0\,. \tag{9}$$

Besides the trivial solutions in which the dilaton and radii are constant, a Kasner-like branch of solutions can be obtained as follows. If the pressures vanish the energy E is conserved. Then (8) can be reduced to an equation just for φ, with general solution

$$\varphi(t) = \log\left[\frac{16\pi^2/E}{t(t+C)}\right] \tag{10}$$

(we have suppressed one constant of integration corresponding to an arbitrary shift in t). One can then integrate the λ_i equations of motion to find

$$\lambda_i(t) = A_i + B_i \log\frac{t}{t+C}\,. \tag{11}$$

The constants of integration A_i are arbitrary, while in order to satisfy the Hamiltonian constraint B_i and C must satisfy $C^2\,(1-\sum_i B_i^2) = 0$. Thus either $C = 0$ and the radii are static, or $\sum_i B_i^2 = 1$ and the radii are time dependent. In both cases, the dilaton rolls monotonically towards weak coupling.

We now turn to the the late-time asymptotic behavior of solutions to the dilaton-gravity equations.[1] First suppose the pressure is negligible, $P_i \approx 0$, as is the case for a universe in equilibrium with all radii sufficiently close to the self-dual radius. At late times the universe will approach the Kasner-like solution (10), (11), with the asymptotic behavior

$$e^{\varphi} \sim \frac{\text{const.}}{t^2} \qquad\qquad e^{\lambda_i} \sim \text{const.} \tag{12}$$

[1] A similar analysis was performed for M-theory in [1].

Thus if $P_i \approx 0$ the dilaton rolls monotonically while the radii approach constants in string frame.[2]

Now suppose that at late times we have m unwrapped dimensions $x^1 \cdots x^m$ and $9 - m$ wrapped dimensions $x^{m+1} \cdots x^9$.[3] Without loss of generality we can go to a T-dual frame where the unwrapped dimensions are all larger than string scale. In this frame we expect that at late times the universe will be dominated by a radiation gas in the unwrapped dimensions; the pressures should vanish in the wrapped dimensions due to a cancellation between winding and KK modes. That is, we expect

$$P_i \sim \begin{cases} e^{-\lambda_i} & i = 1, \ldots, m \\ 0 & i = m+1, \ldots, 9 \end{cases} \tag{13}$$

An ansatz which captures the late-time behavior is

$$e^{\varphi} \sim \frac{1}{t^{\alpha}} \qquad e^{\lambda_i} \sim \begin{cases} t^{\beta} & i = 1, \ldots, m \\ \text{const.} & i = m+1, \ldots, 9 \end{cases} \tag{14}$$

Plugging this ansatz into the equations of motion fixes

$$\alpha = \frac{2m}{m+1} \qquad \beta = \frac{2}{m+1}\,. \tag{15}$$

The dilaton rolls monotonically to weak coupling, while the unwrapped dimensions grow with time and the wrapped dimensions have fixed sizes. Thus if the string winding dynamics in the early universe favors $m = 3$, as suggested by the dimension counting argument reviewed in the introduction, one could naturally explain why three spatial dimensions become large.

3 Equilibrium thermodynamics

In a coupled matter/gravity system the matter energy is determined by the Hamiltonian constraint (5). For stringy matter Tseytlin & Vafa [4] long ago presented a simple picture of the corresponding thermodynamics which is suitable for our purposes. There are two possible phases.

[2]This was also shown in section 5 of [1]. To relate the two solutions note that [1] worked in terms of M-theory time t_M, related to the string-frame time used here by $t_S \sim t_M^{3/4}$.

[3]A dimension x^i is called unwrapped if $\lambda_i > 0$ and the number of winding strings vanishes, or if $\lambda_i < 0$ and the number of momentum modes vanishes.

3.1 Hagedorn phase

Strings have a limiting Hagedorn temperature [26]. For weakly-coupled type II strings the limiting temperature is $T_H = \frac{1}{\pi\sqrt{8}}$. Near this temperature the canonical ensemble fails and one must use the microcanonical ensemble [27]. In the Hagedorn phase the universe contains a dense gas of winding and KK modes. To a good approximation the free energy $F = E - T_H S$ vanishes, so the microcanonical entropy is given by $S(E) = E/T_H$. The total pressure also vanishes, $P_i = -\frac{\partial F}{\partial \lambda_i} = 0$.

The thermodynamics of strings in the Hagedorn phase has been studied by Deo, Jain and Tan [28, 29]. They employ the microcanonical ensemble, with a fixed energy E and a fixed net winding charge in the universe; in a compact space the latter vanishes. They show that the average number of type II strings present with winding charge vector $\mathbf{w}$ and energy ϵ is given by

$$D(\epsilon, \mathbf{w}, E) = \frac{N}{\epsilon} u(\epsilon, E)^{d/2} e^{-u(\epsilon,E)\mathbf{w}^T A^{-1}\mathbf{w}/4} \tag{16}$$

where

$$N = \frac{(2\sqrt{\pi})^{-d}}{\sqrt{\det A}}$$

$$u(\epsilon, E) = \frac{E}{\epsilon(E-\epsilon)}$$

$$A_{ij} = \frac{1}{4\pi^2 R_i^2}\delta_{ij}$$

Here $R_i \equiv e^{\lambda_i}$. As a consistency check, note that the total amount of energy in strings indeed adds up to E:

$$\int_0^E d\epsilon \int d^d w \; \epsilon D(\epsilon, \mathbf{w}, E) = E.$$

We will ignore diagonally-wound topologies (where a string is simultaneously wound on several dimensions) and assume that we have 9 unidimensional string gases. That is, we set $d = 1$ and assume that the total energy available to each dimension is 1/9 of the total energy in the universe. Thus the distribution for a single winding charge w_i is given by

$$D(\epsilon, w_i, E) = \frac{\sqrt{\pi}R}{\epsilon}\sqrt{u(\epsilon, E/9)} \exp\left[-u(\epsilon, E/9) w_i^2 \pi^2 R_i^2\right]. \tag{17}$$

The (thermally averaged) total number of positive windings W_i is then given by

$$\langle W_i \rangle = \int_0^{E/9} d\epsilon \int_0^{\infty} dw_i \; w_i D(\epsilon, w_i, E) = \frac{\sqrt{E}}{12\sqrt{\pi} R_i}. \tag{18}$$

Note that in (18) we are only counting positive winding in some direction; the net winding is zero. The total length of string present is proportional to the energy E, while the physical distribution of string on the torus amounts to a random walk. Thus the average positive winding is simply the average distance from the origin of a random walk. This goes as the square root of the length, or equivalently the square root of the energy.

We have computed the average number of winding modes, but a similar result holds for the average amount of positive KK momentum present, just by replacing $R_i \to 1/R_i$:

$$\langle N_i \rangle = \frac{\sqrt{E} R_i}{12\sqrt{\pi}}. \tag{19}$$

3.2 Radiation phase

Below the Hagedorn temperature the string oscillators make a negligible contribution to the partition function, so we can focus on single-string states which are labeled by an integer-valued momentum vector n_i and an integer-valued winding vector w_i. In the absence of a B-field the string energy levels are

$$\epsilon(\mathbf{n}, \mathbf{w}) = \sqrt{\sum_i \left(\left(\frac{n_i}{R_i} \right)^2 + (w_i R_i)^2 \right)}.$$

The corresponding free energy for a gas of strings is

$$\beta F = 128 \sum_{\mathbf{n} \cdot \mathbf{w} = 0} \log \tanh(\beta \epsilon(\mathbf{n}, \mathbf{w})/2)$$

where we have taken into account that for the type II string we have 128 bosonic and 128 fermionic species of excitations. The condition $\mathbf{n} \cdot \mathbf{w} = 0$ enforces level matching. For dimensions which are large compared to a thermal wavelength we can approximate momentum sums by integrals and neglect winding; likewise for dimensions which are small compared to an inverse thermal wavelength we can approximate winding sums by integrals and neglect momentum. Any remaining intermediate-sized dimensions are frozen, with no excitations. Thus we have an approximate expression for the free energy

$$\beta F \approx 128 \prod_{\text{large}} 2\pi R_i \prod_{\text{small}} \frac{2\pi\alpha'}{R_i} \int \frac{d^d p}{(2\pi)^d} \log\tanh(\beta|p|/2) \tag{20}$$

where d is the total number of unfrozen dimensions. At this point it is convenient to order

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_9| \tag{21}$$

and to define the T-duality invariant 'volume' of the d unfrozen dimensions

$$V_d = \prod_{i=1}^{d} 2\pi e^{|\lambda_i|}\,. \tag{22}$$

Then (20) can be identified with the free energy of a massless ideal gas in d spatial dimensions in a box of volume V_d. To write an equation of state we use the fact that in d spatial dimensions an ideal gas has an energy density $\rho = c_d T^{d+1}$, where (for 128 bosonic and 128 fermionic degrees of freedom)

$$c_d = 128 \cdot \frac{2d!\zeta(d+1)}{(4\pi)^{d/2}\Gamma(d/2)}(2 - 2^{-d})\,. \tag{23}$$

The energy, entropy and total pressure of the gas are given by

$$\begin{aligned} E &= c_d V_d T^{d+1} \\ S &= \frac{d+1}{d} c_d V_d T^d \\ P_i &= -\frac{\partial F}{\partial \lambda_i} = \begin{cases} \operatorname{sign}(\lambda_i) E/d & i = 1, \ldots, d \\ 0 & i = d+1, \ldots, 9 \end{cases} \end{aligned}$$

However we still need to determine d. If the energy is very small then all dimensions are frozen. As the energy increases $\lambda_1, \lambda_2, \ldots$ will successively unfreeze. This means that the temperature in the radiation phase is given by

$$T_{\text{rad}} = \min_k \left(\frac{E}{c_k V_k}\right)^{1/(k+1)}\,. \tag{24}$$

The value of k which minimizes the right hand side is equal to the number of unfrozen dimensions. T_{rad} calculated in this way could be larger than the Hagedorn temperature; this signals that the system is actually in the Hagedorn phase. That is the true temperature of the system is $\min(T_{\text{rad}}, T_H)$.

In the radiation phase we can compute the amount of positive KK momentum present in equilibrium by using the fact that for a one-dimensional massless gas $\langle N_i \rangle$ is related to the pressure by

$$\langle N_i \rangle = \frac{1}{2} P_i R_i. \tag{25}$$

This estimate makes sense for $R_i \gg \sqrt{\alpha'}$, in which case we also have $\langle W_i \rangle = 0$. If on the other hand $R_i \ll \sqrt{\alpha'}$ we just use the T-dual formulas

$$\begin{aligned} \langle W_i \rangle &= -\frac{1}{2} P_i / R_i \\ \langle N_i \rangle &= 0. \end{aligned} \tag{26}$$

4 Winding and KK Annihilations

4.1 Boltzmann equations

If the universe was in equilibrium we could just insert the results of the previous section into the dilaton-gravity equations of motion. But the Brandenberger-Vafa scenario is driven by departures from thermal equilibrium. A crude way to keep track of these departures is to let W_i be the amount of positive winding charge around dimension i. Likewise let N_i be the amount of positive Kaluza-Klein momentum in direction i. Of course since the space is compact there must also be W_i units of anti-winding charge and N_i units of anti-KK-momentum.

Let us, for the time being, assume that each unit of charge is carried by a single string: that is, there are W_i strings each wound once with positive orientation, similarly for the KK modes. We will also, for the time being, assume that the strings have no oscillator excitations. Then the annihilation of these momentum and winding modes is governed by Boltzmann equations, similar to the equations that govern the evolution of M2-brane winding [2]:

$$\begin{aligned} \frac{dN_i}{dt} &= -\frac{\langle f(v)\rangle}{2\pi} e^{\varphi-2\lambda_i}(N_i^2 - \langle N_i\rangle^2) \\ \frac{dW_i}{dt} &= -\frac{\langle f(v)\rangle}{2\pi} e^{\varphi+2\lambda_i}(W_i^2 - \langle W_i\rangle^2) \end{aligned} \tag{27}$$

The cross-section for bosonic wound strings was calculated by Polchinski [30], who found $f(v) = 2/(1-v^2)$ for two anti-parallel strings moving with velocity v. Subsequent studies have evaluated this quantity for F- and D-strings [31]. We will set $f(v) \approx 2$, appropriate for a gas of slowly-moving strings.

Let us make a few comments on the structure of these Boltzmann equations. First, note that they are invariant under T-duality. Second, note that they respect the dimension-counting arguments of Brandenberger & Vafa [3]. An implicit factor of the inverse volume of the universe is present in the definition of e^{φ} (3). But due to the factor $e^{2\lambda_i}$ upstairs in the equation for dW_i/dt, strings wrapped on a large 3-torus will still be able to annihilate effectively, just like particles moving in one large spatial dimension. Finally, we should contrast our Boltzmann equations with the results presented in [9], which were appropriate for strings with a dilaton-independent cross-section such as cosmic strings.

Continuing our study of the Boltzmann equations, we now include string oscillator excitations but still restrict attention to unit winding and momentum charges. With oscillators excited a more accurate cross-section is obtained by replacing[4]

$$\exp(-\lambda_i) \to \epsilon_i$$

in the Boltzmann equation for dN_i/dt, where ϵ_i is the average energy of a string with a unit of KK momentum. Likewise in the dW_i/dt equation we should replace

$$\exp(+\lambda_i) \to \delta_i$$

where δ_i is the average energy of a unit winding string. This is supported by the results of Lizzi and Senda [32], who redo Polchinski's calculation for two highly excited strings – which have many oscillator excitations but no winding – and show that the interaction rate goes like the product of the

[4]We thank R. Myers for bringing this issue to our attention.

energies of the two strings. This modifies the Boltzmann equations to read

$$\begin{aligned}\frac{dN_i}{dt} &= -\frac{1}{\pi}e^{\varphi}\langle\epsilon_i\rangle^2(N_i^2 - \langle N_i\rangle^2)\\ \frac{dW_i}{dt} &= -\frac{1}{\pi}e^{\varphi}\langle\delta_i\rangle^2(W_i^2 - \langle W_i\rangle^2).\end{aligned} \tag{28}$$

We may estimate the typical energy per momentum or winding mode from the distributions given in the previous section:

$$\begin{aligned}\text{Hagedorn phase} \quad &: \quad \langle\epsilon_i\rangle = \frac{E}{9\langle N_i\rangle} \quad \text{(momentum modes)}\\ &\quad\;\; \langle\delta_i\rangle = \frac{E}{9\langle W_i\rangle} \quad \text{(winding modes)}\end{aligned} \tag{29}$$

$$\begin{aligned}\text{Radiation phase} \quad &: \quad \langle\epsilon_i\rangle = 1/R_i \quad \text{(momentum modes)}\\ &\quad\;\; \langle\delta_i\rangle = R_i \quad \text{(winding modes)}\,.\end{aligned}$$

Here we have assumed that in the Hagedorn phase the energy is equally distributed between dimensions. Note that in the Hagedorn phase the average energy per mode scales as $\sqrt{E}$.

Finally we consider strings that are multiply wound around each dimension, so that the number of positively-wound strings and the winding charge W_i are not necessarily the same. We may think of the winding charge as made up of W_i open unit strands that are braided together to form closed strings. Depending on the braiding, there can be anywhere from 1 to W_i closed strings present. Also depending on the braiding, an individual closed string can carry anywhere from 1 to W_i units of winding charge. There are $W_i!$ ways of braiding the strands; we assume all braidings are equally likely. Then the typical strand is part of a closed string that carries winding charge $(W_i + 1)/2$. The cross-section of a string is proportional to its length and hence winding charge, thus we expect the typical string-string cross-section to be enhanced by a factor $((W_i + 1)/2)^2$. Making this modification to the cross-section, and rewriting the Boltzmann equation as an equation for the rate of change of the positive winding charge, a net enhancement factor of $(W_i + 1)/2$ appears on the right hand side relative to (28).

With an analogous modification for multiple-momentum strings the Boltzmann equations read

$$\frac{dN_i}{dt} = -\frac{(N_i+1)}{2\pi}e^{\varphi}\langle\epsilon_i\rangle^2(N_i^2 - \langle N_i\rangle^2) \tag{30}$$

$$\frac{dW_i}{dt} = -\frac{(W_i+1)}{2\pi} e^{\varphi} \langle \delta_i \rangle^2 (W_i^2 - \langle W_i \rangle^2)$$

where the average energies per strand are still given by (29).

One could question our assumption that all braidings are equally likely. Although this seems like a reasonable assumption when the universe is small, on entropic grounds it could be that as some dimensions grow large singly wound strings become favored. Rather than study this issue directly, in our numerical work we will investigate the two extreme possibilities: all strings singly-wound as in (28), or all braidings equally likely as in (30).

4.2 Freeze-out

In an expanding universe the evolution of a species depends on the species' annihilation rate Γ and the cosmological expansion rate (or Hubble parameter) $\dot{R}/R = \dot{\lambda}$. For dilaton gravity one also needs to take into account the rate of change of the dilaton $\dot{\varphi}$. To determine whether a nonzero number of strings survive to the asymptotic future, we need to study how these parameters evolve. If the annihilation rate of wound strings decreases too rapidly it could undermine the naive dimension counting arguments, which implicitly assume that Γ remains non-zero.

To illustrate the possibility of freeze-out consider the following simple situation. At some initial time set all $\lambda_i = 0$. Introduce the same number of unit winding and unit momentum strings in all directions: $N_i = W_i \equiv N$. Suppose further that no oscillators are excited. Then the pressures P_i all vanish, and it is consistent to set the logarithmic scale factors $\lambda_i = 0$ for all time. The remaining equations of motion are very simple. As explained in section 2, the shifted dilaton obeys $\ddot{\varphi} = \frac{1}{2}\dot{\varphi}^2$ with solution

$$e^{\varphi} = \frac{A}{(t-t_0)^2}.$$

Here A and t_0 are two constants of integration. As expected the dilaton rolls monotonically to weak coupling. The Boltzmann equation (for singly-wound strings) reads

$$\frac{dN}{dt} = -\frac{1}{\pi} e^{\varphi} \langle \epsilon \rangle^2 \left(N^2 - \langle N \rangle^2 \right) .$$

To get a feel for whether the strings will freeze out it suffices to set $\langle N \rangle = 0$. Then the general solution is

$$\frac{1}{N(t)} = \frac{1}{N(t_1)} + \frac{1}{\pi} \int_{t_1}^{t} dt' \, e^{\varphi} \langle \epsilon \rangle^2 \,. \tag{31}$$

As long as the integral stays finite as $t \to \infty$ a non-zero fraction of the strings will freeze out. In the case at hand the pressure vanishes, which means the total energy in matter does not change with time; since the radii are fixed the average energy per string $\langle \epsilon \rangle$ also remains constant. Then the integral is strongly convergent, and

$$\frac{1}{N(t)} = \frac{1}{N_\infty} - \frac{A \langle \epsilon \rangle^2}{\pi (t - t_0)} \,.$$

Here we have defined $1/N_\infty \equiv 1/N(t_1) + A\langle \epsilon \rangle^2 / \pi (t_1 - t_0)$. As $t \to \infty$ a non-zero fraction of the unit winding and unit momentum strings do freeze out, with $N(t) \to N_\infty$.

This sort of behavior should be fairly generic, even for solutions that do not sit precisely at the self-dual radius. The pressure vanishes as long as one remains in the Hagedorn phase, giving give rise to a conserved matter energy. Moreover, if the radii change slowly with time, the average energy per string still remains roughly constant. The dilaton, however, will still roll monotonically towards weak coupling, and as long as it does so quickly enough for the integral in (31) to converge, some strings will freeze out. The enhanced cross-section due to multiple winding in (30) does not change this outcome.

This is troubling for the Brandenberger-Vafa scenario, as it shows that simple dimension-counting arguments can fail to capture the true dynamics of winding strings. Our goal in the remainder of this paper is to undertake a detailed numerical investigation of the likelihood of freeze-out.

5 Initial conditions and Holography

We would like to choose initial conditions at random, so as to uniformly sample the possible states of the early universe. In practice we proceed by fixing the initial value of the shifted dilaton φ and the initial volume of the universe V. All other degrees of freedom will be given random initial values, drawn from the probability distribution worked out below.

Ideally, we would average over all possible values of the "coordinates" λ_i, φ together with their canonical momenta using the Liouville measure obtained from the action (2). The microcanonical volume of phase space for dilaton gravity plus matter is

$$\Omega \sim \int d^9\lambda \, d^9\dot{\lambda} \, d\varphi \, d\dot{\varphi} \; e^{-10\varphi} e^S \tag{32}$$

where S is the matter entropy. In the Hagedorn phase this is given by

$$S = E/T_H = (2\pi)^2 e^{-\varphi}(\dot{\varphi}^2 - \sum_i \dot{\lambda}_i^2)/T_H \,. \tag{33}$$

Thus at the level of supergravity the initial conditions which maximize the entropy are

$$\begin{array}{ll} \varphi \rightarrow -\infty & \text{weak string coupling and large volume} \\ \dot{\varphi} \rightarrow -\infty & \text{effective coupling rapidly decreasing} \\ \dot{\lambda}_i = 0 & \text{constant size of torus} \end{array} \tag{34}$$

To set initial conditions we first fix a value of φ. In order for effective supergravity to be valid we must have $e^\varphi \ll 1$. Note that since we're working with effective supergravity, not string theory, only the value of the shifted dilaton matters and we don't need to worry about the underlying dilaton ϕ defined in (3) becoming large. Next we fix a value for $\dot{\varphi}$. For supergravity to be valid we must have $\dot{\varphi} \gtrsim -1$. Then from (33) note that the $\dot{\lambda}$ are Gaussian distributed, with a characteristic spread

$$(\Delta\dot{\lambda}_i)^2 = T_H (2\pi)^{-2} e^\varphi \,. \tag{35}$$

For simplicity we take the $\dot{\lambda}_i$'s to be uniformly distributed about zero, in the interval $[-\sqrt{T_H}(2\pi)^{-1}e^{\varphi/2}, \sqrt{T_H}(2\pi)^{-1}e^{\varphi/2}]$. Note that for $\varphi < 0$ we'll have $-1 < \dot{\lambda}_i < 1$. The scale factors λ_i do not appear in the entropy, so we take them to be uniformly distributed, subject only to the constraint that the T-duality invariant 9-volume V defined in (22) has the specified value: $\sum_i |\lambda_i| = \log(V/(2\pi)^9)$.

Following the M-theory analysis of [2], we can ask about the holographic bound [33, 34, 35, 36]. This is

$$S \le \frac{A_E}{4G} = \frac{2\pi A_E}{\kappa^2} = \frac{2\pi \Omega_8 R_E^8}{\frac{1}{2}(2\pi)^7(\alpha')^4} \tag{36}$$

where Ω_8 is the area of a unit S^8 and the subscript E reminds us that this must be calculated in the Einstein frame. We wish to convert this to string frame, with (note that the regular dilaton, not φ, appears below)

$$R_E = e^{-\phi/4} R_S \tag{37}$$

making

$$S \le \frac{2\Omega_8 \pi^8}{(2\pi)^6} e^{-\varphi-\lambda} \tag{38}$$

on an isotropic torus. Comparing to the initial value we get a bound (again assuming we start in the Hagedorn phase)

$$S = \pi\sqrt{8}(2\pi)^2 e^{-\varphi}\left(\dot{\varphi}^2 - \sum_i \dot{\lambda}_i\right) \le \frac{2\Omega_8\pi^8}{(2\pi)^6} e^{-\varphi-\lambda} \tag{39}$$

$$\pi\sqrt{8}(2\pi)^2\dot{\varphi}^2 \le \frac{2\Omega_8\pi^8}{(2\pi)^6} e^{-\lambda} \tag{40}$$

$$2\pi e^{\lambda} \le \frac{\Omega_8}{128\sqrt{2}\dot{\varphi}^2} \tag{41}$$

We can interpret this as a bound

$$V \sim (2\pi e^{\lambda})^9 \le \left(\frac{\Omega_8}{128\sqrt{2}\dot{\varphi}^2}\right)^9 = \left(\frac{\pi^4}{420\sqrt{2}\dot{\varphi}^2}\right)^9 \tag{42}$$

or equivalently

$$\dot{\varphi}^2 \le \frac{0.16}{V^{1/9}}\,. \tag{43}$$

6 Numerical Analysis

6.1 Initial Conditions

Our simulations proceed by generating multiple sets of initial data, solving the equations of motion numerically, and looking at the number of wrapped dimensions at late times after freeze-out has taken place. In each run we fixed the initial values of V and φ. For the most part we started with $\dot{\varphi} = -1$; this maximizes the entropy while keeping supergravity valid.[5] All other initial conditions are allowed to fluctuate randomly. The $\dot{\lambda}_i$ are chosen from the flat distribution described in the previous section. The initial λ_i are generated by choosing nine random numbers in $[-1, 1]$ and applying an overall scaling so that the initial volume matches the specified value. To assign initial values to N_i and W_i we compute the mean values from section 3 and then add random thermal fluctuations about the mean, of magnitude

$$\Delta N_i \approx \sqrt{\langle N_i \rangle} \qquad \Delta W_i \approx \sqrt{\langle W_i \rangle} \,. \tag{44}$$

Note that we do not impose the holographic bound (43) on our initial data; we have some comments on this below.

To evolve the system we use a Runge-Kutta algorithm. At each time step we begin by computing the total matter energy from the Hamiltonian constraint (5). The equilibrium thermodynamics discussed in section 3 enables us to decide whether the system is in a Hagedorn or radiation phase. Based on this we compute the corresponding thermal expectation values $\langle N_i \rangle$, $\langle W_i \rangle$. We then use the equations of motion to evolve to the next time step.

The dilaton-Einstein equations of motion are given in (6), (7). To solve them we need an expression for the pressures P_i. We set[6]

$$P_i = 2 \left(N_i e^{-\lambda_i} - W_i e^{\lambda_i} \right) \,. \tag{45}$$

[5]Note that we are considering the case where the dilaton is rolling towards weak coupling.

[6]Here we are relating the actual pressures P_i to the actual values of N_i and W_i, allowing for departures from thermal equilibrium. Thus (45) should not be confused with (25), (26) where we used the equilibrium pressures to compute the thermally averaged values of N_i and W_i. Of course for a system in equilibrium the expressions are compatible.

This simple estimate is valid when no oscillators are excited; thus it should be accurate in the radiation phase. In the Hagedorn phase oscillators are excited and the pressure receives corrections. However given the equilibrium values of N_i and W_i in the Hagedorn phase (18), (19) note that on average a cancellation makes the pressure vanish. Thus, although it would be nice to have a more precise expression for the pressure, we do not expect any refinements to (45) to significantly affect our results.

The Boltzmann equations were discussed in section 4. To allow for the effects of multiply-wound strings we ran simulations using two different versions of the Boltzmann equations given in (28) and (30). The first is appropriate for strings that only carry one unit of winding or momentum charge, while the second is appropriate for strings with multiple winding or momentum charges.

In Figure 1 we show what happens when we vary the initial values of φ and V, starting with the initial condition $\dot{\varphi} = -1$. We show the average number of wrapped dimensions present both in the initial configuration and after freeze-out. Clearly a final state with three unwrapped dimensions is not dynamically favored. If one begins at reasonably strong coupling then very few strings are present in the initial state, while if one begins at weak coupling string interactions turn off too rapidly for the required annihilations to occur. In either case three large spatial dimensions is not the most likely late-time geometry.

We have explored what happens if the initial value of $\dot{\varphi}$ is decreased, since the holographic bound (43) restricts the allowed values of this quantity. In Figure 2 we show the initial and final number of wrapped dimensions starting with the initial condition $\dot{\varphi} = -0.15$. The qualitative outcome is the same, just shifted to more negative initial values of φ. This is not surprising, given the Hamiltonian constraint (5): roughly speaking a change in $\dot{\varphi}^2$ can be compensated by shifting φ so as to keep the total energy fixed.

In Figure 3 we show the distribution of initial winding configurations for universes that end up with three large dimensions. Although the number of wrapped dimensions can either increase or decrease with time, it is unlikely that one can begin deep in the Hagedorn phase with nine wrapped dimensions and end up with a three dimensional universe.

In Figure 4 we show how the distribution of final winding configurations depends on the initial value of φ. Although for a rather narrow range of φ three dimensions is the favored outcome, the distribution of final dimensionality is not very sharply peaked.

All results presented so far have been based on the multiply-wound cross section (30). We have studied what happens if we evolve the system using the singly-wound cross section (28). The change in the results is negligible, much less than the widths of the distributions shown in Fig. 4. Thus the qualitative outcome is the same, with no dynamical preference for three dimensions.

7 Conclusions

Our results indicate that – within the context of our approximations – the expansion of the universe has an "all or nothing" character. If initial conditions are such that one begins with many wrapped strings, the strings typically freeze out and keep all dimensions small. On the other hand if one begins with few wrapped strings, the strings typically annihilate and all dimensions decompactify. Between these extremes there are initial conditions that lead to three large dimensions, but such initial conditions are not generic. Fine-tuning the initial conditions to yield three large dimensions is thus possible, but runs counter to the goal of the string gas program: finding a mechanism in which generic initial data yields three large spatial dimensions.

The unexpected chink in the Brandenberger-Vafa scenario we have found is that due to the rolling dilaton the string annihilation cross section becomes weaker than previously realised. To avoid this impasse we would need a mechanism for keeping the string annihilation cross section sufficiently robust.[7] There are ways in which this might be accomplished (e.g. strings wound on finite fundamental groups [6], in confining backgrounds [31], or having unusual kinematic configurations [37]), but as yet none have been studied in adequate detail to determine their viability. Also we should note that, even if one manages to stabilize the cross-section, one would still have to face the issue that the gravitational back-reaction of an anisotropic string gas turns off at late times, due to the factor e^{φ} which appears in the equa-

[7]To study the long-time behavior with an enhanced cross-section one should take thermal fluctuations into account, not only in setting the initial conditions as in (44), but also by adding an explicit noise term to the equations of motion.

tions of motion (7). One might be tempted to postulate a mechanism which stabilizes the dilaton, however this is problematic for reasons discussed in [23]: with pure Einstein gravity strings should freeze out, along the lines of our M-theory analysis [2].

Lest we appear too pessimistic, let us note some directions for future study which might invalidate our conclusions.

- In this paper we have only studied decreasing dilaton solutions, whereas there is also a class of increasing dilaton solutions. If the value of the dilaton grew sufficiently large, the appropriate framework would be M-theory, and the results of our previous paper [2] would apply. However, it is possible that there is an intermediate time in which the dilaton is large enough for string annihilations to be effective, yet small enough for perturbative string theory to be relevant.

- In this paper we made a number of simplifying assumptions. In particular we assumed spatial homogeneity and only considered the radial moduli of the torus. A more complete analysis at the level of effective supergravity would be desirable; steps in this direction have been taken in [19, 21, 24].

The failure of the string gas scenario to naturally lead to three large spatial dimensions may be telling us one of three things. First, perhaps the string gas (or brane gas) framework is supplanted by other dynamics in the early universe, invalidating the approach we have been following. Second, perhaps the measure (32) does not reflect the true distribution of possible initial conditions of the universe. Third, perhaps three spatial dimensions is not favored. If one imagines that many "universes" are created, all with different initial conditions, then some sort of anthropic argument could be invoked. But many people, including ourselves, are uncomfortable with anthropic arguments until every other possibility has been explored. Consequently we intend to return to these cosmological issues as our understanding of string theory in the early universe improves.

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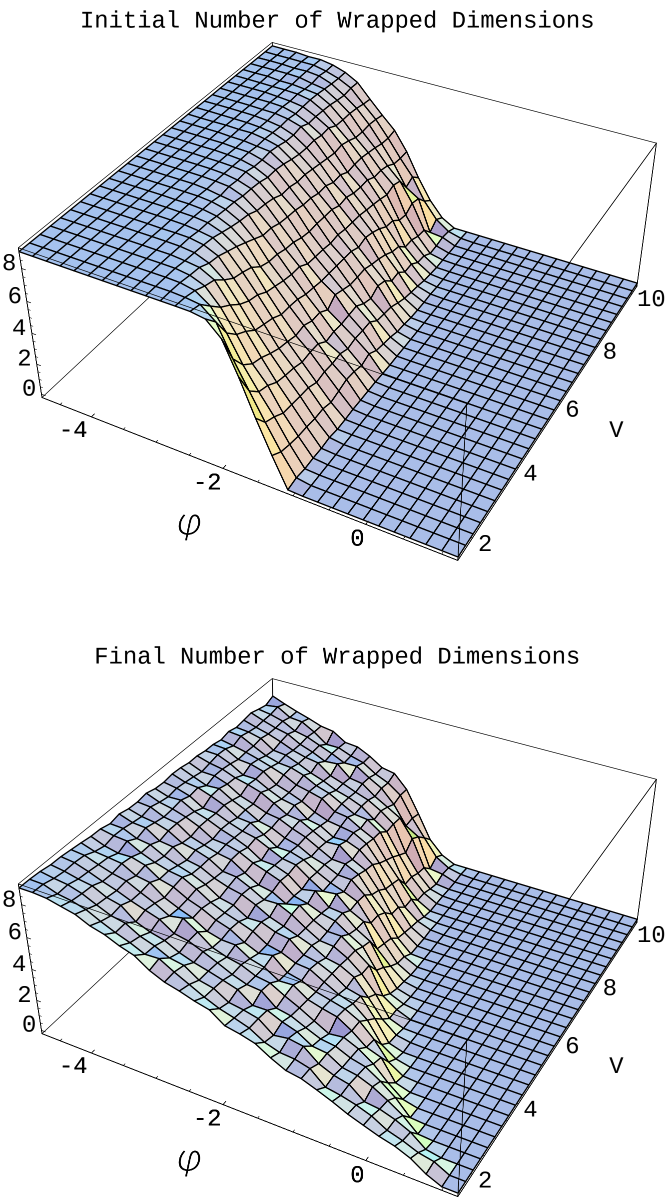

Figure 1: Average initial and final number of wrapped dimensions as a function of the initial coupling and initial volume, starting with $\dot{\varphi} = -1$ and evolved using the multiply-wound cross section. The volume is measured in units of $(2\pi\sqrt{\alpha'})^9$.

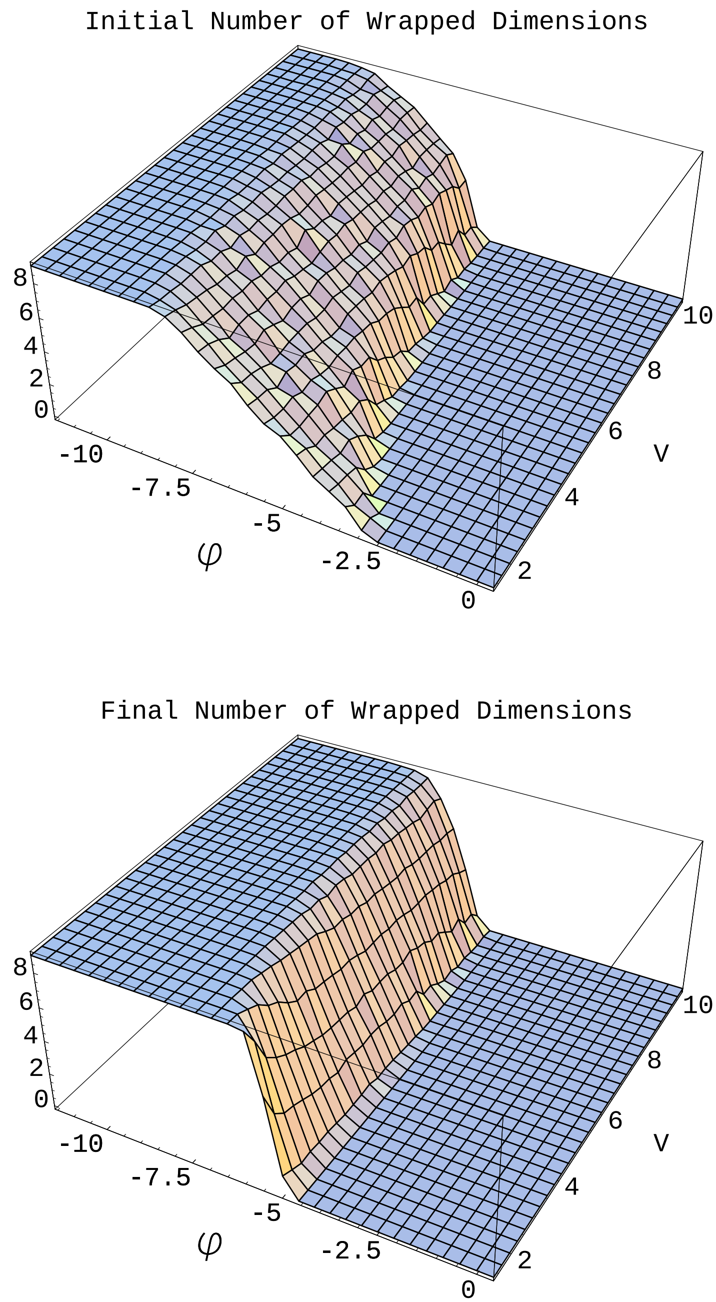

Figure 2: Illustrates the dependence on the initial value of $\dot{\varphi}$. Same as Fig. 1 except the simulations begin with $\dot{\varphi} = -0.15$.

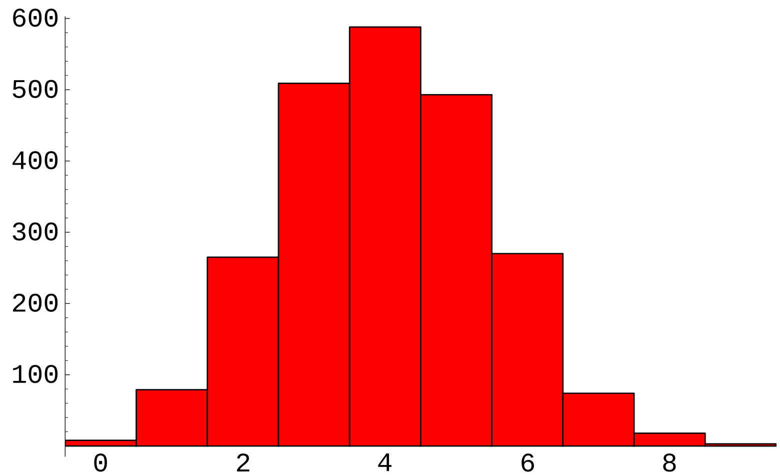

Figure 3: A histogram showing the distribution of the initial number of unwrapped dimensions for universes that end up three dimensional. Extracted from the data set used to generate Fig. 1.

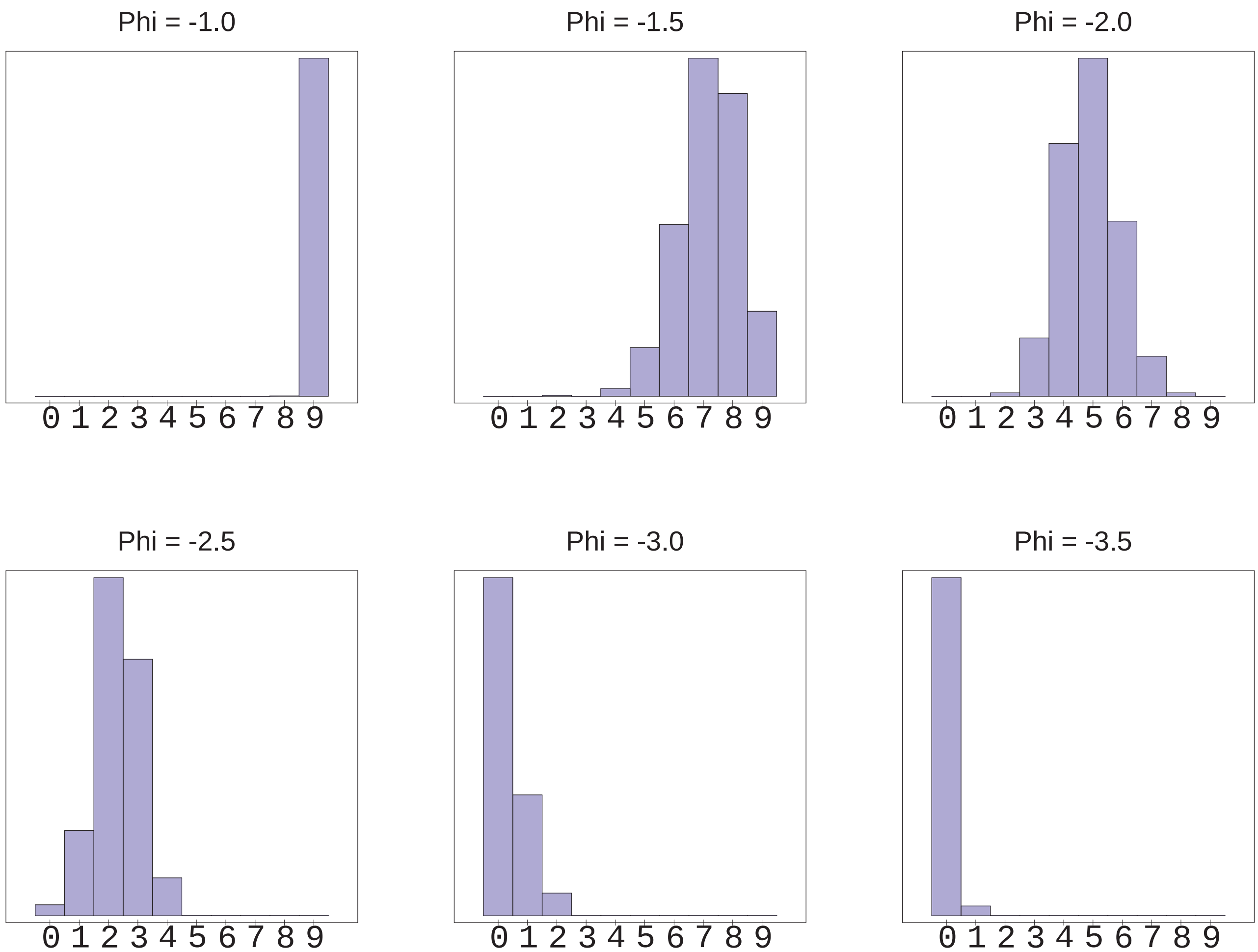

Figure 4: Histograms showing the distribution in the number of unwrapped dimensions at late times for various initial values of φ. Each histogram is based on 10^3 simulations at an initial volume $V = 4.0 \times (2\pi\sqrt{\alpha'})^9$ and an initial $\dot{\varphi} = -1$.