

CU-TP-1067
 UPR-1009-T
 MAD-TH-02-1
 hep-th/0208163

On the Hagedorn Behaviour of PP-wave Strings and $\mathcal{N} = 4$ SYM Theory at Finite R -Charge Density

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Abstract

We discuss the high temperature behaviour of IIB strings in the maximally symmetric plane wave background, and show that there is a Hagedorn temperature. We discuss the map between strings in the pp-wave background and the dual superconformal field theory in the thermal domain. The Hagedorn bound describes a curve in the R -charge chemical potential versus temperature phase diagram of the dual Yang-Mills theory and the theory manifestly exists on both sides. Using a recent observation of Brower, Lowe, and Tan, we update our earlier calculation to reflect that the pp-wave string exists on both sides of the Hagedorn bound as well.

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1 Introduction

For many decades now, string theory has held the promise of answering long standing mysteries that require a consistent theory of quantum gravity. The two most pressing areas are those of black hole physics and cosmology, both of which, in standard treatments, are afflicted by spacetime singularities that conclusively demonstrate the need for a more robust theoretical framework. During the last few years, significant progress has been made in understanding aspects of black hole physics in the context of string theory [1], although the proper resolution of the Schwarzschild black hole singularity remains elusive. More recently, there has also been a renewed surge of activity in applying string theory to issues relevant to cosmology through the advent of brane-world models [2, 3, 4, 5, 6], the focus on de Sitter spacetime [7, 8, 9], the application of holographic ideas [10, 11, 12, 13], and the controversial possibility that string theory (and quantum gravity more generally) might leave an observable imprint in the cosmic microwave background radiation [14, 15, 16, 17, 18, 19, 20].

It has long been recognized that one requisite element of string cosmology is an understanding of string theory at finite temperature, and much work along these lines has been undertaken [21, 22, 23, 24, 25]. These works have also demonstrated that finite temperature studies have the capacity to shed much light on critical, foundational issues of string theory itself. As a prime example, compared with ordinary quantum field theory, string theory is exceedingly well behaved in the ultraviolet, a property largely due to the exponential growth of states as a function of energy. And since the earliest days of string theory it has been recognized that such a growth of states leads to a Hagedorn temperature — a temperature above which the statistical partition function diverges because the exponential Boltzmann suppression is overcome by the exponential density of states. The existence of a Hagedorn temperature is particularly tantalizing because a long-standing, critical question has been: what are the true, fundamental degrees of freedom in string theory? While strings provide a natural and perturbatively useful set of constituents, the existence of a Hagedorn temperature is an indication that strings — somewhat like low energy/temperature hadrons in quantum chromodynamics — may not be the elementary degrees of freedom of string theory. Perhaps, many authors have speculated, just as quark and gluons emerge as the basic ingredients of quantum chromodynamics at high enough temperatures (and energies), the true degrees of freedom of string theory may also emerge at sufficiently high temperatures. In fact, if the specific heat at the Hagedorn temperature is finite, this indicates that the Hagedorn temperature is not a limiting temperature, but, instead, demarcates a phase transition — one that might well signal the appearance of the true, elementary degrees of freedom of string theory.

Over the years there have been a number of studies of the Hagedorn temperature in string theory; most have focused on string propagation in flat backgrounds since the calculation of the density of states requires an exactly soluble model. For example, in ref. [23] open bosonic, open and closed supersymmetric, and heterotic string theories were studied in flat noncompact space and it was found that the Hagedorn temperature is a limiting temperature for the open string models but not for the closed string models. In [24], closed strings were studied on a flat compact manifold (T^9) and it was found that the Hagedorn temperature is a limiting temperature. More recently, [26, 27] have studied D-brane thermodynamics and found, for example, that Dp -branes for $p < 5$ possess a non-limiting Hagedorn temperature

and hence likely have a high temperature Hagedorn phase.

In spite of many studies a precise understanding of the properties that a Hagedorn phase in string theory would exhibit, is still lacking.¹ A number of the works cited did reveal one likely qualitative feature of strings near the Hagedorn temperature — namely, the coalescing of energy into one long string, but very little else is known. In light of the work of [25] this may not be particularly surprising. Through indirect reasoning, these authors showed that the Hagedorn phase is characterized by a drastic reduction in the number of degrees of freedom (the free energy behaves as T^2 instead of T^D) and hence it may well be a highly nontrivial task to identify and rigorously describe the correct high temperature constituents.

In this paper, we undertake a further study of high temperature string theory in light of two recent and related advances. The first advance is the recent realization that string propagation in a maximally supersymmetric pp-wave background provides a non-flat, exactly soluble string theory [29, 30]. It is natural, then, to consider thermal string theory in such a background since we retain the necessary analytical control to make explicit calculations even though the space is not flat. The second advance is the realization that because of the AdS/CFT correspondence, pp-wave string theory is dual to a subsector of a particular Yang-Mills theory [31]. This holds out the promise of a new, dual description of the Hagedorn properties of string theory (in a pp-wave background) and hence the possibility of gaining insight inaccessible by more direct means. In what follows, through direct calculation, we will partially realize these expectations.

In particular, in section 2 we will lay out the basic formalism for explicit calculations of thermal strings in a pp-wave background. In section 3 we will discuss the high temperature behaviour of strings in a pp-wave background, and will show that, indeed, there is a Hagedorn temperature. Close examination of the thermal properties of strings near the Hagedorn temperature will reveal it to be a limiting temperature.² This is counter to the intuition that at high energies the pp-wave string only differs mildly from the Minkowski string. Therefore, it appears that we cannot gain much insight into the nature of a Hagedorn phase (and the fundamental degrees of freedom of string theory at high temperature) by studying strings in the pp-wave background. However, in section 4 we study the dual description of these results in $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory, and obtain a somewhat unexpected result. Extending the map between pp-waves and the dual conformal field theory given in [31] to the thermal domain, we find that the string Hagedorn bound is a function of the super-Yang-Mills temperature and R -charge chemical potential. Surprisingly, the known regimes of the dual Yang-Mills theory at finite temperature and R -charge density lie beyond the Hagedorn bound, and hence are inaccessible to the thermal pp-wave string. This suggests that the limiting Hagedorn temperature may not be absolute, and that there exists a physical regime on the other side of the bound.³ To further motivate this conjecture, we note that thermodynamics in the context of quantum gravity is notoriously subtle, and it is questionable whether the ideal gas approximation — the approximation we use in our calculations — is a valid one, as in this approximation one effectively tunes the coupling to

¹Interesting recent progress on this question is reported in [28].

²Recent work of [55] modifies some of the conclusions presented here. Extending our results (by analyzing full thermodynamic integrals rather than just their integrands as done here) [55] shows that the Hagedorn temperature rather signals a phase transition.

³In fact, it turns out that this conjecture is true [55]; see note added in proof for details.

zero. Perhaps beyond the ideal gas approximation, the Hagedorn temperature ceases to be limiting and the pp-wave string probes the full phase diagram of the SYM theory. In section 5 we offer our conclusions and directions for further work.

While completing this paper, the preprint [54] appeared, which, in the revised version, contains the same results on the Hagedorn temperature and limiting behaviour of the IIB pp-wave string as we find in our sections 2 and 3.

2 Thermodynamics of pp-wave strings

2.1 Partition function in a pp-wave background

Our understanding of statistical mechanics, as of quantum field theory, is ultimately based on a Hamiltonian formulation of the theory. As such statistical mechanics suffers from the same complications as quantum field theory when one considers the system in a space-time background which is not flat. Only when the background space-time has a timelike Killing vector, can we at present make sense of the notion of a Hamiltonian as the generator of time-evolution and construct an associated quantum field theory or statistical partition function.⁴

A first requirement, therefore, is to analyze the symmetries of the pp-wave background. These have been extensively discussed in [32, 33]. The maximally symmetric pp-wave background for the IIB string, to which we will limit our attention, has the metric

$$ds^2 = -dx^+ dx^- - \mu^2 z_i^2 (dx^+)^2 + dz_i^2 \quad ; \quad i = 1, \dots, 8 \quad (2.1)$$

with the curvature supported by a constant five-form flux

$$F_{+1234} = \frac{\mu}{g_s}. \quad (2.2)$$

The symmetries of the metric are two translational isometries in the dx^\pm directions, a rotational $SO(8)$ symmetry, broken to $SO(4) \times SO(4)$ by the presence of the five-form flux, and eight Lorentz-boost-like rotations in the z^i, x^- plane. In addition there is the Lorentz boost $x^+ \rightarrow \alpha x^+, x^- \rightarrow x^-/\alpha$, which is only a symmetry when accompanied by the rescaling $\mu \rightarrow \mu/\alpha$.

Consider now an arbitrary linear combination of the translational isometries

$$\xi(a, b) = a \frac{\partial}{\partial x^+} + b \frac{\partial}{\partial x^-}. \quad (2.3)$$

We may compute its nature, and construct a time-like Killing vector, according to which we quantize the system. The length squared of $\xi(a, b)$ is

$$||\xi||^2 = -ab - \mu^2 z_i^2 a^2 \quad ; \quad (2.4)$$

hence $\xi(a, b)$ is timelike if $-a(b + \mu^2 z_i^2 a) < 0$. For instance, the naive choice $a = b = 1$, in which case the Killing vector is simply $\partial/\partial t$, ($x^\pm = t \pm r$), obeys this inequality. The

⁴Thermodynamics is strictly defined in the infinite volume limit, which for string theory, or any quantum theory of gravity, does not exist due to Jeans' instability. For sufficiently small string coupling this problem may be ignored, which is the point of view we shall take. We also assume that use of the canonical ensemble is valid.

dependence of the length of $\xi(a, b)$ on the position z simply reflects the redshift of temperature for various classes of observers.⁵ We will choose both $a > 0$ and $b > 0$ to ensure that the Killing vector is everywhere timelike. Quantizing the pp-wave system with such a $\xi(a, b)$ as Hamiltonian, the pp-wave partition function is formally

$$\mathcal{Z}(a, b; \mu) = \text{Tr}_{\mathcal{H}} e^{-bp_- - ap_+} \quad (2.5)$$

with the standard relation $p_{\pm} = -i\partial/\partial x^{\pm}$. We have denoted the implicit dependence on the curvature of the pp-wave background by the variable μ .

The symmetry of the theory under Lorentz scaling plus boost implies that the partition function (2.5) only depends on dimensionless combinations of a , b and μ . This still means that the partition function is a function of *two* variables, rather than a single temperature. The pp-wave geometry is not isotropic and these two parameters correspond geometrically to the length of the time-like Killing vector and its angle with respect to an arbitrarily chosen fiducial tangent vector field. The first selects the family of observers and the second the temperature of the heat-bath this family measures. In flat space the first variable, the angle, is irrelevant, as isotropy assures that all families of observers are equivalent. The identical expression for the partition function eq. (2.5) in flat space, where $\mu = 0$, can be Lorentz-boosted to a standard one-parameter partition function with temperature $T^{-1} = \sqrt{ab}$. The Lorentz-boost is a symmetry of the flat space, $\mu = 0$ theory, and the flat space partition function computed before or after the boost will give the same answer. The pp-wave geometry does not possess this symmetry, and the partition function (2.5) will therefore be a two-parameter function of a and b . In section 4 we will see that the dependence on the second parameter corresponds to the inclusion of a chemical potential in the dual CFT partition function.⁶

Let us conclude with a final comment. The pp-wave metric is not static. Hence straight Euclideanization of the spacetime will cause the metric to be complex. This does not mean that we cannot discuss thermal physics. As emphasized at the beginning of this section, to construct a partition function, the metric only needs to possess a timelike Killing vector to serve as Hamiltonian — the spacetime must be stationary. In that case the concept of thermal equilibrium can make sense. If spacetime is also static, constructing a path integral from the statistical partition function will be equivalent to compactifying Euclidean time. If the spacetime is not static, one has to resort back to the Hamiltonian approach, as we do here.

2.2 Single and multi-string partition functions from the lightcone

As is well known, the partition function of an ideal gas of IIB pp-wave strings is a weighted product of the partition function of a single IIB pp-wave string (see e.g. [22, 23, 24])

$$\ln \mathcal{Z}(a, b; \mu) = \sum_{r=1}^{\infty} \frac{1}{r} \left(Z_1^B(ar, br; \mu) - (-1)^r Z_1^F(ar, br; \mu) \right) . \quad (2.6)$$

⁵These are hypothetical non-geodesic observers located at fixed z , which are the natural ones to consider in the formal construction of the statistical partition function as a weighted distribution of Hamiltonian eigenvalues. Physical observers would follow geodesics instead.

⁶We thank Jan de Boer for pointing this out to us.

This relation holds for a system of weakly interacting strings immersed in a heat bath of temperature $T^{-1}(z) = \sqrt{ab + \mu^2 z^2 a^2}$. By definition the bosonic and fermionic components of the single string partition function are

$$Z_1^B(\beta) = \frac{1}{2} \text{Tr}(1 + (-1)^{F_{st}}) e^{-\beta H} , \quad (2.7)$$

$$Z_1^F(\beta) = \frac{1}{2} \text{Tr}(1 - (-1)^{F_{st}}) e^{-\beta H} , \quad (2.8)$$

where F_{st} is the spacetime fermion number operator. The partition function is more conveniently expressed in terms of the building blocks

$$Z_1^{Thermal}(\beta) = \text{Tr} e^{-\beta H} , \quad Z_1^{WI}(\beta) = \text{Tr}(-1)^{F_{st}} e^{-\beta H} . \quad (2.9)$$

For a spacetime supersymmetric string, as we have in mind, the Witten index $Z_1^{WI}(\beta)$ counts the number of bosonic minus fermionic zero energy groundstates. This is a finite number (it equals unity, see [34]) and gives only a small temperature independent correction to the free energy. As we will be interested in the high temperature behaviour of the pp-wave string, we will ignore this contribution.

It remains for us to determine the single string partition function (dropping the *Thermal* superscript),

$$Z_1(a, b; \mu) = \text{Tr}_{\mathcal{H}} e^{-bp_- - ap_+} , \quad (2.10)$$

for the IIB pp-wave string. This is straightforwardly achieved, as it is known how to quantize the IIB pp-wave string in the lightcone gauge [29]. In this gauge, worldsheet diffeomorphism invariance is used to set $X^+(\sigma, \tau) = x^+ + (2\pi\alpha' p^+) \tau$, and the theory is divided into sectors of fixed $p_- = p^+$. After solving for the Virasoro constraints, the remaining transverse degrees of freedom are eight *free* worldsheet supersymmetric scalar multiplets with mass $m = \mu\alpha' p^+$ and $p_+ = (2\pi)^{-1} \int d\sigma P_+$ as lightcone Hamiltonian;

$$p_+ \equiv H_{lc}^{(m)} = \frac{1}{\alpha' p_-} \left(\omega_0(N_0^B + N_0^F) + \sum_{n \geq 1} \omega_n^{(m)}(N_n^B + N_n^F + \tilde{N}_n^B + \tilde{N}_n^F) \right) \quad (2.11)$$

with

$$\omega_n^{(m)} = \text{sign}(n) \sqrt{n^2 + m^2} . \quad (2.12)$$

$N_n^{B,F}$ and $\tilde{N}_n^{B,F}$ are the number operators for the eight transverse right- and left-moving bosonic and fermionic oscillator modes, and the right- and leftmoving zero modes are identified as usual. As the (lightcone) theory is worldsheet supersymmetric the zero point energy cancels between the bosons and fermions.

The choice of lightcone gauge solves the two worldsheet reparametrization constraints explicitly. One single consistency condition remains to be imposed. This is the descendant of the relation

$$X^-(\sigma, \tau) = X^-(\sigma + 2\pi, \tau) \Leftrightarrow \alpha_0^- = \tilde{\alpha}_0^- \quad (2.13)$$

and becomes the level-matching constraint on the transverse oscillators. The geometric reason behind this constraint is, as always, the circle isometry of the worldsheet. The generator of worldsheet translations in the sigma directions, normalized to integer eigenvalues, is formally unchanged and equals $P = \sum_{n \geq 1} n(N_n - \tilde{N}_n)$. Imposing this constraint on the space of states by the introduction of a delta function, the single string partition function equals

$$Z_1(a, b; \mu) = \int_0^\infty dp_- \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 e^{-bp_-} z_{lc}(\tau_1, \frac{a}{2\pi\alpha'p_-}; m = \mu\alpha'p_-) , \quad (2.14)$$

where $z_{lc}(\tau_1, \tau_2; \mu\alpha'p_-)$ is the lightcone partition function

$$z_{lc}(\tau_1, \tau_2; m) \equiv \text{Tr}_{states} e^{-2\pi\tau_2(\alpha'p_- H_{lc}^{(m)}) + 2\pi i\tau_1 P} . \quad (2.15)$$

Note that we have chosen to define the lightcone partition function such that the mass $m = \mu\alpha'p_-$ and the modular parameter $\tau_2 = a/2\pi\alpha'p_-$ are independent. The steps to evaluate eq. (2.15) are familiar. The bosonic and fermionic sectors separate and each can be expressed in terms of the building blocks [34] (see also [35])

$$\Theta_{\alpha,\delta}(\tau, \bar{\tau}; m) = e^{4\pi\tau_2 E_c^\delta(m)} \prod_{n=-\infty}^{\infty} (1 - e^{-2\pi\tau_2|\omega_{n+\delta}| + 2\pi i\tau_1(n+\delta) + 2\pi i\alpha})(1 - e^{-2\pi\tau_2|\omega_{n-\delta}| + 2\pi i\tau_1(n-\delta) - 2\pi i\alpha}) , \quad (2.16)$$

where we have chosen as argument the standard complex combination of the modular parameters $\tau = \tau_1 + i\tau_2$. Note that despite this notation the building blocks are not analytic in τ . The argument of the exponential prefactor, E_c^δ , is the zero point energy shift or Casimir energy for a chiral (one quarter of a complex two-dimensional) boson of mass m with periodicity $\phi(\sigma + 2\pi, t) = e^{2\pi i\delta}\phi(\sigma, t)$,

$$\begin{aligned} E_c^\delta(m) &= \frac{1}{4} \int_{-\infty}^{\infty} dk \omega(k) - \frac{1}{4} \sum_{n=-\infty}^{\infty} |\omega_{n\pm\delta}| \quad ; \quad \omega(k) = \sqrt{k^2 + m^2} \\ &= -\frac{1}{2\pi^2} \sum_{p=1}^{\infty} \int_0^\infty ds e^{-p^2 s - \frac{(\pi m)^2}{s}} \cos(2\pi\delta p) . \end{aligned} \quad (2.17)$$

In terms of these building blocks the lightcone partition function with the value of τ appropriate for eq. (2.14), $\tau = \tau_1 + ia/2\pi\alpha'p_-$, equals

$$z_{lc}(\tau, \bar{\tau}; \mu\alpha'p_-) = \int d\lambda dm \left[\frac{\Theta_{\frac{1}{2},0}(\tau, \bar{\tau}; m)}{\Theta_{0,0}(\tau, \bar{\tau}; m)} \right]^4 e^{2\pi i\lambda(m - \mu\alpha'p_-)} . \quad (2.18)$$

The integration over the Lagrange multiplier λ enforces the proper value for the mass m . Inserting this expression into eq. (2.6) and changing integration variables to $\tau_2 = ar/2\pi\alpha'p_-$ we find the following expression for the free energy of an ideal gas of pp-wave strings

$$\begin{aligned} -\beta F &= \ln \mathcal{Z}(a, b; \mu) \\ &= -\frac{a}{2\pi\alpha'} \int d\lambda dm \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int_0^\infty d\tau_2 \frac{1}{\tau_2^2} \left[\frac{\Theta_{\frac{1}{2},0}(\tau, \bar{\tau}; m)}{\Theta_{0,0}(\tau, \bar{\tau}; m)} \right]^4 \sum_{r=\text{odd}}^\infty e^{-\frac{abr^2}{2\pi\alpha'\tau_2} + 2\pi i\lambda(m - \mu ar/2\pi\tau_2)} . \end{aligned} \quad (2.19)$$

3 High temperature behaviour of pp-wave strings

Having computed the partition function of the IIB pp-wave string, we may start to analyze its thermodynamic properties. Particular questions of importance are:

- Does the IIB pp-wave string have a Hagedorn temperature (do level densities grow exponentially)?
- If so, does the Hagedorn behaviour of the IIB signal the onset of a phase transition, or is there a limiting temperature?

As the difference between the IIB pp-wave string and the Minkowski string is rather mild, we expect that the gross characteristics are unchanged. Naively both questions should therefore receive a positive answer, and the spectrum of the IIB pp-wave string appears to support this conclusion. For highly excited modes with individual frequency $\omega_n = \sqrt{n^2 + (\mu\alpha'p_-)^2} = n(1 + \frac{(\mu\alpha'p_-)^2}{2n^2} + \dots)$ a given (quantized) energy level $E = N$ is somewhat less easily partitioned, but asymptotically (where μ can effectively be ignored) it certainly approaches the exponentially growing Minkowski spectrum. The pp-wave density of states is therefore also expected to grow exponentially, albeit less rapidly at high T than the flat space string. There should therefore exist a pp-wave Hagedorn temperature, whose value is somewhat higher than that of the Minkowski space string.

By the same reasoning, one is inclined to associate the Hagedorn temperature with a phase transition and not with a limiting temperature. The parameter which determines the nature of the Hagedorn behaviour, limiting or phase transition, is the subleading power γ in the asymptotic density of states [27]

$$d(E \rightarrow \infty) \sim E^\gamma e^{E/T_H} . \quad (3.1)$$

Again, one expects that asymptotically the effect of the pp-wave deformation, characterized by the value of μ , on the coefficient γ is negligible. Surprisingly we will find that this is not the case. A closer look reveals why. The effective mass in the lightcone gauge is $m = \mu\alpha'p_-$ rather than μ . Although asymptotic level densities $d(n)$ are clearly unaffected by μ , the density of states $d(E)$ at high energies (equivalent to large lightcone momentum p_-) may not be insensitive to the deformation. We will show that the effect of μ remains mild in that the growth of states is still exponential — there is a Hagedorn temperature — and just mild enough so that the pp-wave string also undergoes a phase transition at $T = T_H$, as recently shown by [55].⁷

3.1 Hagedorn behaviour

Let us first show that there is a Hagedorn temperature at which the free energy diverges. This happens when the integrand of eq. (2.19) diverges exponentially for $\tau_2 \rightarrow 0$. Hence we need to compute the asymptotic behaviour of the building blocks $\Theta_{\alpha,\delta}(\tau, \bar{\tau}; m)$ as $\tau \rightarrow 0$.

⁷In a preprint version, our calculations suggested that the Hagedorn temperature was limiting, a result we conjectured to be an artifact of approximations being made. Indeed, as the recent new results of [55] show, this is the case; see endnote for details.

As we already briefly noted, this has to be done with some care as the effective mass m is constrained to equal $m = \mu ar/2\pi\tau_2$; see eq. (2.19). Even so, the effect of μ on the asymptotic level densities of the string, we have argued, should be mild, and we therefore make the ansatz that for $\tau_2 \rightarrow 0$, with $m\tau_2 = \tilde{\mu}$ fixed, the building blocks $\Theta_{\alpha,\delta}(\tau, \bar{\tau}; m)$ diverge exponentially;

$$\Theta_{\alpha,\delta}(\tau, \bar{\tau}; \frac{\tilde{\mu}}{\tau_2}) \sim \exp \left[\frac{\xi(\tilde{\mu})}{\tau_2} \right] \quad \text{as } \tau_2 \rightarrow 0. \quad (3.2)$$

To compute the coefficient $\xi(\tilde{\mu})$, consider the logarithm of eq. (2.16)

$$\ln \Theta_{\alpha,\delta}(\tau, \bar{\tau}, \frac{\tilde{\mu}}{\tau_2}) = 4\pi\tau_2 E_c^\delta(\frac{\tilde{\mu}}{\tau_2}) + \sum_{n=-\infty}^{\infty} \ln(1 - e^{-2\pi\tau_2|\omega_{n+\delta}| + 2\pi i\tau_1(n+\delta) + 2\pi i\alpha}) + \text{c.c.} \quad (3.3)$$

The coefficient $2\pi\tau_2|\omega_{n+\delta}|$ equals

$$2\pi\tau_2|\omega_{n+\delta}| = 2\pi\tilde{\mu} \sqrt{1 + \frac{(n+\delta)^2\tau_2^2}{\tilde{\mu}^2}}. \quad (3.4)$$

In the limit $\tau_2 \rightarrow 0$ we may replace the sum over n by an integral, provided we keep the ratio $\tau_1/\tau_2 \equiv \theta$ fixed (The limit is therefore equal to $|\tau| \rightarrow 0$, $\arg(\tau)$ fixed. If we keep τ_1 finite instead, the limit $\tau_2 \rightarrow 0$ is convergent).

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \ln(1 - e^{-2\pi\tilde{\mu}\sqrt{1+(n+\delta)^2\tau_2^2/\tilde{\mu}^2} + 2\pi i\theta\tau_2(n+\delta) + 2\pi i\alpha}) &\xrightarrow{\tau_2 \rightarrow 0} \\ \frac{\tilde{\mu}}{\tau_2} \int_{-\infty}^{\infty} dn \ln(1 - e^{-2\pi\tilde{\mu}\sqrt{1+n^2} + 2\pi i\tilde{\mu}\theta n + 2\pi i\alpha}) &\equiv -\frac{\tilde{\mu}}{|\tau|} f(\tilde{\mu}, \theta, \alpha), \end{aligned} \quad (3.5)$$

and we have determined the coefficient of divergence $\xi(\tilde{\mu})$ implicitly in terms of an unknown function $f(\tilde{\mu}, \theta, \alpha)$

$$\xi(\tilde{\mu}) = -\frac{\tilde{\mu}\tau_2}{|\tau|} [f(\tilde{\mu}, \theta, \alpha) + \bar{f}(\tilde{\mu}, \theta, \alpha)]. \quad (3.6)$$

That the divergence of the building blocks $\Theta_{\alpha,\delta}(\tau, \bar{\tau}; m)$ depends on the direction with which one approaches $\tau = 0$, is reflective of their non-analyticity.

This function $f(\tilde{\mu}, \theta, \alpha)$ is formally equal to

$$\begin{aligned} f(\tilde{\mu}, \theta, \alpha) &= \frac{|\tau|}{\tau_2} \sum_{\ell=1}^{\infty} \int_{-\infty}^{\infty} dn \frac{e^{-2\pi\ell\tilde{\mu}\sqrt{1+n^2} + 2\pi i\ell\tilde{\mu}\theta n + 2\pi i\ell\alpha}}{\ell} \\ &= 2 \sum_{\ell=1}^{\infty} \frac{e^{2\pi i\ell\alpha}}{\ell} K_1(2\pi\ell\tilde{\mu}\sqrt{1+\theta^2}), \end{aligned} \quad (3.7)$$

where $K_1(x)$ is the modified Bessel function of the second kind [36]. For $\alpha = 0$, $f(\tilde{\mu}, \theta, 0) = f(\tilde{\mu}\sqrt{1+\theta^2}, 0)$ is a monotonic function which for small and large values of its argument

asymptotes to

$$\begin{aligned}
f(x, 0) &\equiv 2 \sum_{\ell=1}^{\infty} \frac{K_1(2\pi\ell x)}{\ell} , \\
x \rightarrow 0 & : \quad f(x, 0) \sim \frac{\pi}{6x} , \\
x \rightarrow \infty & : \quad f(x, 0) \sim \frac{e^{-2\pi x}}{\sqrt{x}} .
\end{aligned} \tag{3.8}$$

For $\alpha = \frac{1}{2}$, $f(x, \frac{1}{2})$ behaves as

$$\begin{aligned}
f(x, \frac{1}{2}) &\equiv 2 \sum_{\ell=1}^{\infty} (-1)^\ell \frac{K_1(2\pi\ell x)}{\ell} , \\
x \rightarrow 0 & : \quad f(x, \frac{1}{2}) \sim -\frac{\pi}{12x} , \\
x \rightarrow \infty & : \quad f(x, \frac{1}{2}) \sim -\frac{e^{-2\pi x}}{\sqrt{x}} .
\end{aligned} \tag{3.9}$$

Note that $\bar{f}(x, \alpha) = f(x, -\alpha) = f(x, -\alpha + 1)$.

Collecting this information, we establish the Hagedorn temperature as the combination of parameters a, b which fail to dampen the exponential growth of oscillator states in the UV. Integrating out the Lagrange multiplier λ in eq. (2.19) and substituting the proper value for the mass m , the exponential growth equals

$$\left[\frac{\Theta_{\frac{1}{2}, 0}(\tau, \bar{\tau}; \frac{\mu ar}{2\pi\tau_2})}{\Theta_{0, 0}(\tau, \bar{\tau}; \frac{\mu ar}{2\pi\tau_2})} \right]^4 \xrightarrow{\tau_2 \rightarrow 0} \exp \left[\frac{8\mu ar}{2\pi|\tau|} \left(f\left(\frac{\mu ar}{2\pi} \sqrt{1+\theta^2}, 0\right) - f\left(\frac{\mu ar}{2\pi} \sqrt{1+\theta^2}, \frac{1}{2}\right) \right) \right] . \tag{3.10}$$

Because $f(x, \alpha)$ tends very rapidly to zero for increasing x , the leading divergent term in the integrand of eq. (2.19) is due to the single string contribution for which $r = 1$. Thus for small values of τ_2 the integrand is dominated by an exponential factor

$$\exp\left(-\frac{ab}{2\pi\alpha'\tau_2}\right) \exp \left[\frac{8\mu a}{2\pi|\tau|} \left(f\left(\frac{\mu a}{2\pi} \sqrt{1+\theta^2}, 0\right) - f\left(\frac{\mu a}{2\pi} \sqrt{1+\theta^2}, \frac{1}{2}\right) \right) \right] , \tag{3.11}$$

and the partition function will be convergent if the parameters a, b which determine the temperature are chosen such that the argument in (3.11) is negative. This is only the case when

$$b > 8 \frac{\tau_2 \mu \alpha'}{|\tau|} \left(f\left(\frac{\mu a}{2\pi} \sqrt{1+\theta^2}, 0\right) - f\left(\frac{\mu a}{2\pi} \sqrt{1+\theta^2}, \frac{1}{2}\right) \right) . \tag{3.12}$$

The explicit $\theta \sim \tau_1$ dependence may appear curious, but, as we mentioned previously, this is a consequence of the non-analytic nature of the building blocks. In the final expression for the free energy τ_1 is integrated over and no dependence remains. To explicitly perform the τ_1 integral is not straightforward. Inspection of the RHS of eq. (3.12), however, shows

that it is a monotonically decreasing function of θ . This allows us to argue that a minimum requirement for convergence is eq. (3.12) with θ set to zero,⁸

$$b > 8\mu\alpha' \left(f\left(\frac{\mu a}{2\pi}, 0\right) - f\left(\frac{\mu a}{2\pi}, \frac{1}{2}\right) \right). \quad (3.13)$$

As expected the pp-wave string therefore has a Hagedorn temperature, which occurs when this inequality is saturated. The physical temperature measured by different observers (parameterized by a) at the bound equals

$$T_H(a; z)^{-2} = 8a\mu\alpha' \left(f\left(\frac{a\mu}{2\pi}, 0\right) - f\left(\frac{a\mu}{2\pi}, \frac{1}{2}\right) \right) + \mu^2 z^2 a^2. \quad (3.14)$$

For $\mu = 0$, this indeed gives the standard, observer independent, IIB Hagedorn temperature

$$ab = T_H^{-2} = 8\pi\alpha' \left(\frac{\pi}{6} + \frac{\pi}{12} \right) = 4\pi^2\alpha', \quad (3.15)$$

whereas for $\mu \neq 0$ the rapid vanishing of $f(x)$ implies that the Hagedorn temperature rises quickly with μ , but only disappears for μ strictly infinite.

Note finally that in all the expressions above only those combinations of a , b and μ appear which are invariant under the Lorentz boost plus rescaling: $a \rightarrow \alpha a$, $b \rightarrow b/\alpha$ and $\mu \rightarrow \mu/\alpha$, as we argued in section 2.

3.2 Phase transition or limiting temperature

The crucial question is whether the Hagedorn temperature T_H is limiting or whether the divergence of the partition function for $T > T_H$ signals the onset of a phase transition. The distinction between the two modes of behaviour is whether thermodynamic quantities at $T = T_H$ are finite or not. We will focus on the free energy, $\beta F = -\ln \mathcal{Z}$, eq. (2.19). A necessary, but not sufficient, requirement for the possibility of a phase transition is that the free energy is finite at $T = T_H$. To establish whether this is the case, we also need to know the subleading divergence of the building blocks $\Theta_{\alpha,\delta}(\tau, \bar{\tau}; m)$ as $\tau_2 \rightarrow 0$. Postulating that the behaviour of $\Theta_{\alpha,\delta}(\tau, \bar{\tau}; m)$ is of the form

$$\Theta_{\alpha,\delta}(\tau, \bar{\tau}; \frac{\tilde{\mu}}{\tau_2}) \sim \tau_2^{c(\tilde{\mu})} e^{\frac{\xi(\tilde{\mu})}{\tau_2}}, \quad (3.16)$$

the free energy, eq. (2.19), will be finite at $T = T_H$ if $c(\mu, \alpha = \frac{1}{2}) - c(\mu, \alpha = 0)$ is large enough to overcome the τ_2^{-2} factor in the measure.⁹ This is the case if $c(\mu, \alpha = \frac{1}{2}) - c(\mu, \alpha = 0) > \frac{1}{2}$. The naive expectation — which we will show *not* to be true — is that $c(\mu)$ is a smooth function with $c(\mu = 0; \alpha \neq 0) = 0$; $c(\mu = 0; \alpha = 0) = -1/2$.

⁸For an important subtlety regarding this reasoning, see note added in proof.

⁹Applying these observations to determine the nature of the Hagedorn temperature depends sensitively on working with the full integral instead of just the integrand of the partition function; see endnote. As [55] shows, the τ_1 Lagrange multiplier integral contributes an additional factor $\tau_2^{3/2}$ to the measure. Hence if $c(\mu, \alpha = \frac{1}{2}) - c(\mu, \alpha = 0) > -\frac{1}{8}$, the free energy is finite and the Hagedorn bound demarcates a phase transition.

We can compute the power c by taking the logarithm as before,

$$\ln \Theta_{\alpha,\delta}(\tau, \bar{\tau}; \frac{\tilde{\mu}}{\tau_2}) - \frac{\xi(\tilde{\mu})}{\tau_2} = c(\tilde{\mu}) \ln \tau_2 + \mathcal{O}(\tau_2) . \quad (3.17)$$

Recalling the determination of $\xi(\tilde{\mu})$ from eq. (3.5), the LHS of this equation equals

$$\sum_{n=-\infty}^{\infty} \ln(1 - e^{-2\pi\tilde{\mu}\sqrt{1+(n+\delta)^2\tau_2^2/\tilde{\mu}^2+2\pi i\theta\tau_2(n+\delta)+2\pi i\alpha}}) - \frac{\tilde{\mu}}{\tau_2} \int_{-\infty}^{\infty} dn \ln(1 - e^{-2\pi\tilde{\mu}\sqrt{1+n^2+2\pi i\tilde{\mu}\theta n+2\pi i\alpha}}) .$$

Setting $\theta = 0$ (basing ourselves on the arguments below eq. (3.12)¹⁰) and using the Poisson resummation identity

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dx e^{2\pi i k x} f(x) , \quad (3.18)$$

this can be simplified to

$$2 \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} dn \ln(1 - e^{-2\pi\tilde{\mu}\sqrt{1+n^2\tau_2^2/\tilde{\mu}^2+2\pi i\alpha}}) e^{2\pi i k(n-\delta)} = c \ln \tau_2 + \mathcal{O}(\tau_2) . \quad (3.19)$$

Formally the LHS equals

$$-2 \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{e^{2\pi i(\ell\alpha-k\delta)}}{\ell} \frac{\tilde{\mu}}{\tau_2} \frac{K_1(2\pi\ell\tilde{\mu}\sqrt{1+(k/\ell\tau_2)^2})}{\sqrt{1+(k/\ell\tau_2)^2}} . \quad (3.20)$$

Evaluating eq. (3.20) in the limit $\tau_2 \rightarrow 0$, we may approximate the Bessel function $K_1(2\pi\ell\tilde{\mu}\sqrt{1+(k/\ell\tau_2)^2})$ with its asymptote at infinity,

$$\tau_2 \rightarrow 0 : \quad \text{LHS}_{(3.19)} = -2 \sum_{k,\ell=1}^{\infty} \frac{\tilde{\mu} e^{2\pi i(\ell\alpha-k\delta)}}{\sqrt{\ell^2\tau_2^2 + k^2}} \frac{e^{-2\pi\ell\tilde{\mu}\sqrt{1+(k/\ell\tau_2)^2}}}{\sqrt{4\ell\tilde{\mu}\sqrt{1+(k/\ell\tau_2)^2}}} \rightarrow 0 , \quad (3.21)$$

which goes to zero exponentially fast. We find *no* logarithmic divergence and conclude that $c(\tilde{\mu}) = 0$ for all values of α and δ . This establishes our claim. For the implication is that the free energy is infinite at $T = T_H$ and there is no phase transition, contrary to our intuition based on the flat space, $\mu = 0$ results.¹¹

This answer is surprising, however. How does it mesh with the known result for $\mu = 0$? Inspection shows that in eq. (3.20) the limits $\mu \sim \tilde{\mu} \rightarrow 0$ and $\tau_2 \rightarrow 0$ do not commute. If we take the limit $\tilde{\mu} \rightarrow 0$ first, we may approximate the Bessel function $K_1(2\pi\ell\tilde{\mu}\sqrt{1+(k/\ell\tau_2)^2})$ with its asymptote near zero, instead of the asymptote near infinity

$$\tilde{\mu} \rightarrow 0 : \quad \text{LHS}_{(3.19)} = -2 \sum_{k,\ell=1}^{\infty} \frac{\tilde{\mu} e^{2\pi i(\ell\alpha-k\delta)}}{\sqrt{\ell^2\tau_2^2 + k^2}} \frac{1}{2\pi\ell\tilde{\mu}\sqrt{1+(k/\ell\tau_2)^2}} . \quad (3.22)$$

¹⁰This is allowed, if we correctly account for any contributions to the τ_2 measure, see footnote 9.

¹¹See footnote 9. With the improved treatment of the τ_1 integration of [55], one concludes that the Hagedorn bound demarcates a phase transition.

The dependence on $\tilde{\mu}$ drops out, and one is tempted to conclude that $c(\tilde{\mu} = 0)$ also vanishes. However, iff $\alpha = 0$ and $\delta = 0$, the resulting double sum is divergent. One is not allowed to take the limit $\tilde{\mu} \rightarrow 0$ within the sum for $\alpha, \delta = 0$, and it follows that our unstated assumption that $c(\tilde{\mu}; \alpha, \delta = 0)$ is a smooth function of $\tilde{\mu}$ is incorrect. A careful analysis, which we have included in an appendix for completeness, yields the known value $c(\tilde{\mu} = 0; \alpha, \delta = 0) = -\frac{1}{2}$.

This incompatibility of limits is already evident in the expression for the frequency $\omega_n = \sqrt{n^2 + (\mu\alpha p_-)^2}$. For $\mu \neq 0$ and only for $\mu \neq 0$, in the high energy limit $p_- \rightarrow \infty$, all oscillator levels can be populated at no cost in extra energy. Pursuing the argument put forth in the beginning of this section to the end, one would conclude that this costless contribution from the tower of string states gives rise to a divergent free energy. Hence, the Hagedorn temperature is a limiting temperature.¹²

We thus see that the Hagedorn behaviour of the $\mu = 0$ IIB Minkowski string is fundamentally different from that of the IIB pp-wave string; there is no continuum limit between the two. There is in fact a good physical explanation for why our intuition that at high energies the μ dependence of the background is irrelevant, and that the presence of μ should only affect the IR behaviour, fails. String theory is modular invariant, which means that UV physics “knows” about IR physics. This also holds for the pp-wave string. Indeed, “strikingly”, despite their nonanalyticity, the building blocks $\Theta_{\alpha,\delta}(\tau, \bar{\tau}; \frac{\tilde{\mu}}{|\tau|})$ have nice modular properties [34, 35]. Under the S -modular transformation, which mixes UV and IR physics, they transform as

$$\Theta_{\alpha,\delta}(\tau, \bar{\tau}; \frac{\tilde{\mu}}{|\tau|}) = \Theta_{-\delta,\alpha}(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}, \tilde{\mu}) . \quad (3.23)$$

The exact asymptotics of $\Theta_{-\delta,\alpha}(\tau, \bar{\tau}; \tilde{\mu})$ at $\tau_2 \rightarrow \infty$, including any power law divergence, are easily read off from the defining expression (2.16) (based on our previous arguments, τ_1 is again set to zero from here on)

$$\lim_{\tau_2 \rightarrow \infty} \Theta_{-\delta,\alpha}(\tau, \bar{\tau}; \tilde{\mu}) = \exp(4\pi\tau_2 E_c^\alpha(\tilde{\mu})) . \quad (3.24)$$

Hence the exact $\tau_2 \rightarrow 0$ divergence of $\Theta_{\alpha,\delta}(\tau, \bar{\tau}; \frac{\tilde{\mu}}{\tau_2})$ equals

$$\lim_{\tau_2 \rightarrow 0} \Theta_{\alpha,\delta}(\tau, \bar{\tau}; \frac{\tilde{\mu}}{\tau_2}) = \exp(4\pi \frac{E_c^\alpha(\tilde{\mu})}{\tau_2}) . \quad (3.25)$$

This is indeed equal to the expression we had before (eq. (3.6)), and illustrates how the high energy physics knows about the infrared: the coefficient of exponential growth is the 2D Casimir energy. This unambiguously determines that the power law divergence coefficient $c(\mu \neq 0) = 0$.

The essence of why this does not match with the flat space result $c(\mu = 0; \alpha = 0) = -\frac{1}{2}$, resides in the modular properties of $\Theta_{0,0}(\tau, \bar{\tau}; m)$. This building block strictly vanishes in the limit $\mu \rightarrow 0$. As shown in [35], the modular properties of the corresponding flat space building block, the Dedekind eta function, which include a power law factor, are obtained at the subleading level in $\mu = 0$ from the modular properties of the building block $\Theta_{0,0}(\tau, \bar{\tau}; m)$.

¹²This intuitive picture is naive. As we conjecture later on, and [55] has recently been able to show, the Hagedorn bound ought to, and does, signal a phase transition.

The vanishing of $\Theta_{0,0}(\tau, \bar{\tau}; m)$ for $\mu \rightarrow 0$ is due to the appearance of zero-modes. Naively at high energies the presence or absence of zero modes, an infrared issue, should not matter. What the computation explicitly shows, is that the effective mass in the problem is $m = \tilde{\mu}/\tau_2$, rather than μ . Thus in the naive UV, $\tau_2 \rightarrow 0$, the mass blows up, and cannot be ignored.¹³ This is the reason why the Hagedorn behaviour of the pp-wave string is different from that of the flat space string.¹⁴

4 Implications for the CFT dual

Pp-wave strings can be derived as a scaling limit from AdS strings, and through this limit they inherit a duality map with a subsector of a CFT [31]. They are the only known example of a gauge-theory/string theory duality where the string side is understood, and as such hold promise in both explaining what stringy effects are in gauge theory, as well as using gauge theory to answer string theory questions. For the IIB pp-wave string the precise map is to a double scaling limit of $\mathcal{N} = 4$, $SU(N)$ Super-Yang-Mills (SYM), the BMN limit, which sends both the rank of the gauge group N and the R -charge J to infinity, keeping the ratio J^2/N fixed [31, 37, 38, 39, 40, 41, 42, 43].

The symmetry charges of both theories are mapped to each other as follows

$$\frac{2p_+}{\mu} = E - J \quad , \quad 2\mu\alpha'p_- = \frac{J}{\sqrt{\lambda}} \quad , \quad \lambda = g_{YM}^2 N = 4\pi g_s N \quad . \quad (4.1)$$

Here E is the energy of the theory on S^3 (or conformal weight on R^4), and J a generator of a $U(1)_R$ subgroup of $SU(4)_R$. The string partition function, $\mathcal{Z}(a, b; \mu)$ we computed in eq. (2.19), is a function of the two independent variables, a , b , thermodynamically conjugate to the operators p_+ , p_- (recall that μ can be removed by the Lorentz boost). One linear combination of these is the temperature of the string heat bath, the other designates the family of observers which perform the measurement. From the SYM point of view we see that we are computing the partition function at a finite temperature T_{YM} and finite R charge chemical potential ν_R . To determine the precise relation between a , b and T_{YM} and ν_R , we need to recall some facts about both theories and the map in more detail.

The derivation of the duality between pp-wave strings and the BMN doubly scaling limit of $\mathcal{N} = 4$ SYM starts with a global coordinate choice for the AdS factor. This means we are considering the SYM theory on S_3 rather than \mathbb{R}^4 . The finite temperature, finite density SYM partition function is therefore well defined, with the scale set by the radius of the S_3 . A second detail concerns thermodynamics in a system with non-zero groundstate energy. When we consider a system (ideal gas) at finite temperature, what we really mean is that the constituents of the system have an average *excitation* energy $E_{ex} \sim T$ above the groundstate. In other words the temperature is the thermodynamic conjugate to the

¹³Oddly enough, it is the S -transformed part of the building block $\tau_2 \rightarrow \infty$, the naive IR region, where the effect of the mass μ is the mildest.

¹⁴For clarity, the issue we are referring to here is how in string theory IR zero modes impact the UV Hagedorn behaviour and the different zero mode structures of the pp-wave string compared to the Minkowski string theory. The pp-wave string behaves as a T^8 compactification. Whether the Hagedorn temperature is limiting or signals a phase transition, depends on one further step of analysis.

excitation energy $E_{ex} = E - E_0$ rather than the absolute energy E (see e.g. [44]). In $\mathcal{N} = 4$ SYM a state with a specific R -charge J has at least an absolute energy J due to the BPS condition. Hence within a sector of fixed J , there is a groundstate energy, and when considering the thermodynamics within such a sector, the temperature is the conjugate variable to $E - J$. The corresponding statement on the pp-wave string side is of course that the light-cone Hamiltonian p_+ is positive semi-definite.

With this in mind, and the map of operators eq. (4.1), the identification of the string thermodynamic variables in terms of those of $\mathcal{N} = 4$ SYM is straightforward. The temperature and chemical potential equal

$$T_{YM}^{-1} = \frac{a\mu}{2} , \quad \sqrt{\lambda}\nu_R = \frac{b}{2\mu\alpha'} . \quad (4.2)$$

However, we are not considering the full $\mathcal{N} = 4$ $SU(N)$ SYM theory but only the physical degrees of freedom that survive the double scaling limit $N, J \rightarrow \infty$, with J^2/N fixed. It is therefore more appropriate to define a chemical potential $\tilde{\nu} \equiv \nu_R \sqrt{\lambda}$ conjugate to $\tilde{J} \equiv J/\sqrt{\lambda}$. Finally, all computations in the preceding sections are done in the ideal gas approximation where $g_s \ll 1$. We should therefore keep in mind that the results obtained should only apply when the Yang-Mills coupling is also vanishingly weak.

4.1 Hagedorn bound for $\mathcal{N} = 4$ SYM

We have seen that for an ideal gas of IIB pp-wave strings the free energy diverges at

$$\frac{b}{8\mu\alpha'} = f\left(\frac{a\mu}{2\pi}, 0\right) - f\left(\frac{a\mu}{2\pi}, \frac{1}{2}\right) . \quad (4.3)$$

When the left-hand side is less than the right-hand side, the (perturbative) IIB pp-wave string theory does not exist, as the Hagedorn behaviour is limiting.¹⁵

In terms of SYM variables the Hagedorn bound reads

$$\tilde{\nu} = 4 \left(f\left(\frac{1}{\pi T}, 0\right) - f\left(\frac{1}{\pi T}, \frac{1}{2}\right) \right) . \quad (4.4)$$

In figure 1, we have plotted the bound in the temperature-chemical potential phase diagram of $\mathcal{N} = 4$ SYM.¹⁶ The existence of a limiting Hagedorn temperature in the dual pp-wave string would appear to say that the area of the phase diagram where the BMN density $\tilde{\nu}$ is less than the RHS of eq. (4.4) is inaccessible. This is peculiar, as it includes the familiar $\tilde{\nu} \sim \nu = 0$ region of the theory. In fact through the AdS/CFT correspondence quite a lot has become known about the finite density, finite temperature behaviour of large N , $\mathcal{N} = 4$ SYM through the study of spinning branes in supergravity [45, 46, 47, 48, 49, 50]. Unfortunately these studies are not able to clarify what the physics is near the Hagedorn bound. They are performed in supergravity, the $\alpha' \rightarrow 0$ limit of the AdS string theory, and form a complementary regime to that of the pp-wave [39].

¹⁵This is the conclusion we drew in a preprint version. However, as shown in footnote 9 and the note added in proof, using the improved calculation of [55] the Hagedorn temperature is seen to demarcate a phase transition both for compactifications on T^p , $p < 9$ and pp-wave strings, updating our and previous results.

¹⁶Timelikeness of the Killing vector also constrains the variables a and b . For a global Killing vector $a, b > 0$ simply implies that the SYM temperature and density are positive semi-definite.

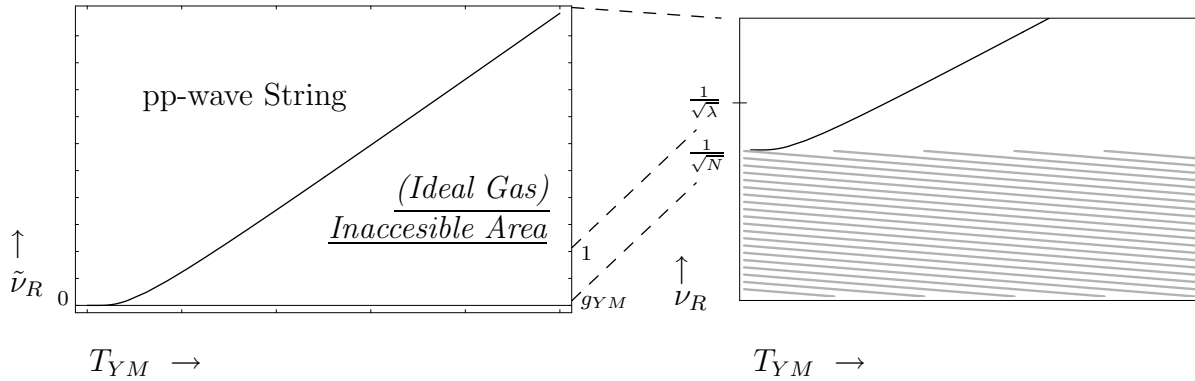


Figure 1: Temperature-chemical potential plot for $g_{YM}^2 \ll 1$ of the BMN limit of $\mathcal{N} = 4$ SYM on S_3 . The right-hand-side shows where the $g_{YM}^2 \ll 1$ BMN limit region lies within the full temperature-density plot of $\mathcal{N} = 4$ SYM. The shaded region, where ν is not parametrically comparable to $1/\sqrt{N}$ is excluded. This includes the giant graviton phase where $\nu \sim \sqrt{\lambda}/N$ [39].

On the other hand the region of applicability of these studies is beyond the Hagedorn bound and therefore, surely, this region is physically accessible. A possible explanation for the apparent mismatch between the IIB pp-wave string theory and $\mathcal{N} = 4$ SYM is that extending the usual zero temperature duality may be more subtle than the approach used in this section.¹⁷ Perhaps, for example, it is only within the ideal gas approximation of vanishingly weak coupling that the Hagedorn bound is limiting. Indeed with small, but non-zero coupling, an infinite free energy in a gravitational system is not likely to be a stable situation. At finite $g_s \lesssim 1$ the system will fall out of equilibrium long before one approaches the Hagedorn bound. One suspects that at finite string coupling, multistring bound states and/or nonperturbative states, such as D-brane black holes, become dominant at high energies, whereas in the ideal gas approximation the high energy behavior is effectively that of a single string. Perhaps, then, the Hagedorn bound reflects a breakdown of the duality between *ultra perturbative* (perturbative string theory in the ideal gas approximation) IIB pp-wave string theory and $\mathcal{N} = 4$ SYM.¹⁸

5 Conclusion

We have studied the finite temperature behavior of IIB strings in a pp-wave background and found, as expected from flat-space experience, a Hagedorn temperature. However, whereas the Hagedorn temperature in Minkowski space indicates the onset of a phase transition, our

¹⁷Throughout our analysis we have worked in the context of the canonical ensemble; it would be worth checking that temperature fluctuations are under control to ensure the validity of this assumption. The recent paper [55] discusses this point.

¹⁸This conjecture of ours, that the Hagedorn bound should demarcate a phase boundary, has indeed been shown correct by Brower, Lowe and Tan [55]. They are able to come to this conclusion by including contributions to the integrand of the partition function arising from the Lagrange multiplier τ_1 integral; see note added in proof.

calculations show the ideal gas pp-wave Hagedorn temperature to have a different character: it is a limiting temperature.¹⁹ At first this might seem surprising since one would have thought that the pp-wave spacetime deformation becomes irrelevant at high energies; the high temperature pp-wave string should be smoothly connected to the high temperature Minkowski space string by the limit $\mu \rightarrow 0$. Yet, in the context of string theory, and finite temperature string theory in particular, this limit has to be taken with care; as we have seen here, the nature of the Hagedorn temperature has a nontrivial dependence on μ .

Naively, large scale modifications to space-time do not affect ultraviolet physics. But in the example studied here, as in many other string theoretic situations, this reasoning fails; perhaps this could have been expected here from the fact that the scalar curvature diverges at $z = 0$. Certainly within string theory, modular invariance implies that high energy physics is dependent on the two-dimensional worldsheet Casimir energy, whereas one would have expected the Casimir energy to be more relevant for light string excitations. Indeed, one way to interpret the Hagedorn temperature in string theory is as the appearance of a massless winding mode in the timelike direction.²⁰ The fact that strings simultaneously know about the large and small scale structure means that only in very controlled situations the high energy behavior is that of the Minkowski string. The calculations here indicate that in situations where the 2D Casimir energy is affected by the background this is not the case.

The dual description, using the BMN limit of $\mathcal{N} = 4$ SYM, adds another interesting layer to the analysis as the dual theory exists for parameters which appear to be forbidden on (ideal gas) string theory arguments.²¹ The simplest explanation is that the duality, at least in the form used here, is breaking down. There is also the possibility that the dual theory is telling us that somehow — notwithstanding the results we have found — the Hagedorn temperature on the pp-wave string side is not strictly limiting and can be crossed. This is tantalizing because for string theories such as the IIB Minkowski string in which the Hagedorn divergence is associated with a phase transition, the identification of the degrees of freedom in the Hagedorn phase is one of the important outstanding questions [25] (see [28] for the most recent progress). Perhaps, with further study beyond the ideal gas approximation, the existence of a dual description for pp-wave string theory may give insight into this critical issue.

We conclude with a question. The traditional interpretation of the Hagedorn temperature, by analogy with QCD, is a confinement/deconfinement transition. Such a transition is known to occur for large N $\mathcal{N} = 4$ SYM on S_3 when the temperature T is of order one in units of the S_3 radius [51], although it is unclear whether this transition is related to Hagedorn behaviour of the AdS string (see, however, [52]). It would be interesting to see if the results in [51] can be extended to include finite R -charge density as well. Perhaps this might shed more light on the large R -charge density - finite temperature behaviour of $\mathcal{N} = 4$ SYM.

Note added in proof: As this article was being prepared for publication, a preprint by

¹⁹See footnotes 15 and 18.

²⁰From this viewpoint, the results of [53], which studies T-duality along the timelike Killing vector, may be interesting.

²¹The conjecture that the Hagedorn behaviour should signal a phase transition, which we partly based on this observation, has been found true [55]; see note added in proof.

Brower, Lowe and Tan appeared which improved the analysis of the asymptotic behaviour of the partition function. For the value of the Hagedorn temperature, their observations have no effect. However, the nature of the transition is affected. Brower, Lowe and Tan's improved treatment of the $\theta \sim \tau_1$ Lagrange multiplier integral found that it contributes an extra factor of $\sqrt{\tau_2}$ to the remaining τ_2 integral. Combined with a Jacobian factor τ_2 from changing to $(\tau_2, \theta = \tau_1/\tau_2)$ coordinates this means that the measure in the final τ_2 integral equals $d\tau_2/\sqrt{\tau_2}$ rather than $d\tau_2/\tau_2^2$. This implies that free energy is finite at T_H and that the Hagedorn bound may demarcate a phase transition for pp-wave strings. Similarly Brower, Lowe and Tan's results indicate that closed strings compactified on T^p have a non-limiting Hagedorn temperature for $p < 9$. Our conjecture that the Hagedorn temperature should demarcate a phase transition is thereby borne out, and this opens up a new avenue to explore both the Hagedorn phase of string theory and the connection between pp-wave strings and $\mathcal{N} = 4$ super-Yang-Mills at finite chemical potential. We are grateful to R. Brower, D. Lowe, and C. Tan for comments and discussions.

Acknowledgments: We thank Maulik Parikh for collaboration in the early stages of this work and many extensive and intensive discussions. We also thank David Berman, Jan de Boer, Chong-Sun Chu, Clifford Johnson, Dan Kabat, Al Mueller, Hiroshi Ooguri, Dam Thanh Son, Matt Strassler, and Bo Sundborg for fruitful discussions and suggestions. KS and GS would like to thank the organizers and participants of the Amsterdam Summer workshop on String Theory and BG and GS would like to thank the Aspen Center for Physics for providing stimulating environments to pursue this research. ISCAP gratefully acknowledges the generous support of the Ohrstrom foundation. KS and BG gratefully acknowledge support from the DOE grant DE-FG-02-92ER40699. GS gratefully acknowledges support from the DOE grant EY-76-02-3071, and the University of Pennsylvania SAS Dean's funds.

A Subleading divergence of the Building Blocks

We seek the coefficient of the $\ln(\tau_2)$ divergence in eq. (3.20)

$$-2 \sum_{k, \ell=1}^{\infty} e^{2\pi i(\ell\alpha - k\delta)} \frac{\tilde{\mu} K_1\left(\frac{2\pi\tilde{\mu}}{\tau_2} \sqrt{(\ell\tau_2)^2 + k^2}\right)}{\sqrt{(\ell\tau_2)^2 + k^2}} + \text{c.c.} \stackrel{\tau_2 \rightarrow 0}{=} c(\tilde{\mu}, \alpha, \delta) \ln(\tau_2) + \dots \quad (\text{A.1})$$

in the limit $\tau_2 \rightarrow 0$ for $\tilde{\mu} = 0$. Compared to eq. (3.20) in the text, we have added the complex conjugate contribution to the RHS; recall from eq. (2.16) and eq. (3.3) that the building blocks $\Theta_{\alpha, \delta}(\tau, \bar{\tau}, m)$ are real. This double sum behaves differently in this limit for $\alpha \neq 0$ or $\delta \neq 0$ and $\alpha, \delta = 0$. The symmetry in the sums over ℓ and k allows us to concentrate only on the case where either $\alpha \neq 0$ or $\delta \neq 0$. We will choose the former.

Noting that the expression can be rewritten so that the summation parameters, ℓ and k , always multiply $\tilde{\mu}$, we may approximate for $\tilde{\mu}$ small the double sum by a double integral. This has to be done with care since the summation is from 1 to infinity. The resulting double integral equals

$$-2 \int_{\tilde{\mu}}^{\infty} dk \int_{\tilde{\mu}\tau_2}^{\infty} d\ell (e^{2\pi i \frac{\alpha}{\tilde{\mu}\tau_2} \ell} + e^{2\pi i \frac{\alpha}{\tilde{\mu}\tau_2} \ell}) \frac{K_1\left(\frac{2\pi}{\tau_2} \sqrt{\ell^2 + k^2}\right)}{\tau_2 \sqrt{\ell^2 + k^2}}. \quad (\text{A.2})$$

We are integrating over the first quadrant of the k, ℓ plane except for a rectangle centered on the origin with area $4\tilde{\mu}^2\tau_2$. As μ is very small, we approximate the lower boundary by a circle of the same area, and change to polar coordinates. We obtain

$$-2 \int_{\tilde{\mu}\sqrt{\frac{2\tau_2}{\pi}}}^{\infty} r dr \frac{K_1\left(\frac{2\pi r}{\tau_2}\right)}{\tau_2 r} \int_0^{\pi/2} d\theta \left(e^{2\pi i \frac{r\alpha \cos \theta}{\tilde{\mu}\tau_2}} + e^{-2\pi i \frac{r\alpha \cos \theta}{\tilde{\mu}\tau_2}} \right). \quad (\text{A.3})$$

The angular integral yields a Bessel function of the first kind, and after a rescaling of r one finds

$$- \int_{\tilde{\mu}\sqrt{\frac{8\pi}{\tau_2}}}^{\infty} dr K_1(r) J_0\left(\frac{r\alpha}{\tilde{\mu}}\right). \quad (\text{A.4})$$

The integral (A.4) is convergent for $\tilde{\mu} = 0$ at the upper boundary, but not at the lower boundary. The result depends on whether α does or does not vanish: the limits $\alpha \rightarrow 0$, $\tilde{\mu} \rightarrow 0$ do not commute.

If $\alpha \neq 0$, we may approximate the Bessel function $J_0(r\alpha/\tilde{\mu})$ for $\tilde{\mu} = 0$ with its asymptote at infinity

$$x \rightarrow \infty : \quad J_0(x) \rightarrow \sqrt{\frac{2}{\pi x}} \cos(x - \pi/4), \quad (\text{A.5})$$

and obtain

$$- \int_{\tilde{\mu}\sqrt{\frac{8\pi}{\tau_2}}}^{\infty} dr K_1(r) J_0\left(\frac{r\alpha}{\tilde{\mu}}\right) = - \int_{\tilde{\mu}\sqrt{\frac{8\pi}{\tau_2}}}^{\infty} dr K_1(r) \sqrt{\frac{2\tilde{\mu}}{\pi r \alpha}} \cos\left(\frac{r\alpha}{\tilde{\mu}} - \pi/4\right). \quad (\text{A.6})$$

Substituting at the lower boundary the asymptote of $K_1(x)$ near $x = 0$,

$$x \rightarrow 0 : \quad K_1(x) \sim \frac{1}{x}, \quad (\text{A.7})$$

one finds that the behaviour of the integral near the lower boundary equals

$$\begin{aligned} - \int_{\tilde{\mu}\sqrt{\frac{8\pi}{\tau_2}}}^{\Lambda} dr \frac{1}{r} \sqrt{\frac{2\tilde{\mu}}{\pi r \alpha}} \cos\left(\frac{r\alpha}{\tilde{\mu}} - \pi/4\right) &= - \sqrt{\frac{2\tilde{\mu}}{\pi \alpha}} \cos(\pi/4) \left[-\frac{2}{\sqrt{r}} \right]_{\tilde{\mu}\sqrt{\frac{8\pi}{\tau_2}}}^{\Lambda} \\ &\simeq \sqrt{\frac{4}{\pi \alpha}} \cos(\pi/4) \left(\frac{\tau_2}{8\pi} \right)^{1/4} + f(\Lambda) + \mathcal{O}(\tilde{\mu}). \end{aligned} \quad (\text{A.8})$$

The small $\tilde{\mu}$ behaviour is finite, and there is no logarithmic divergence for $\tau_2 \rightarrow 0$. Instead the integral vanishes as a quarter power. Note, however, that this power has a multiplicative constant which is inversely proportional to α . The answer therefore only holds if $\alpha \neq 0$.

For $\alpha = 0$ we should approximate the Bessel function $J_0(x)$ with its asymptote near zero rather than infinity. This is just unity. Again approximating $K_1(x) \rightarrow 1/x$ by its small x asymptote we now find logarithmic behaviour at the lower boundary

$$\begin{aligned} - \int_{\tilde{\mu}\sqrt{\frac{8\pi}{\tau_2}}}^{\Lambda} dr \frac{1}{r} &= \ln\left(\tilde{\mu}\sqrt{\frac{8\pi}{\tau_2}}\right) + \mathcal{O}(\tilde{\mu}) + f(\Lambda) \\ &= -\frac{1}{2} \ln \tau_2 + \dots \end{aligned} \quad (\text{A.9})$$

We read off the expected result $c(\tilde{\mu} = 0) = -\frac{1}{2}$.

Hence the building blocks $\Theta_{\alpha,\delta}(\tau, \bar{\tau}, m = 0)$ have a subleading power law divergence for $\tau_2 \rightarrow 0$,

$$\Theta_{\alpha,\delta}(\tau, \bar{\tau}; 0) \sim \tau_2^{c(\alpha,\delta)} e^{\frac{\xi(\tilde{\mu}=0)}{\tau_2}}, \quad (\text{A.10})$$

with $c(\alpha \neq 0 \text{ or } \delta \neq 0) = 0$; $c(\alpha, \delta = 0) = -\frac{1}{2}$, as was known by other methods.

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