

# Bouncing and cyclic string gas cosmologies

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We show that, in the presence of a string gas, simple higher-derivative modifications to the effective action for gravity can lead to bouncing and cyclic cosmological models. The modifications bound the expansion rate and avoid singularities at finite times. In these models the scale factors can have long loitering phases that solve the horizon problem. Adding a potential for the dilaton gives a simple realization of the pre-big bang scenario. Entropy production in the cyclic phase drives an eventual transition to a radiation-dominated universe. As a test of the Brandenberger-Vafa scenario, we comment on the probability of decompactifying three spatial dimensions in this class of models.

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# 1 Introduction

A fundamental question in cosmology is whether the universe has always existed, or whether it came into being a finite time in our past. It could be that the age of the universe is finite; at the classical level the singularity theorems of general relativity make such an assumption seem unavoidable [1]. The other possibility is that the universe has infinite age. A number of past-eternal models have been developed, exploiting the fact that quantum effects or other modifications to general relativity can get around the singularity theorems [2].

String theory should ultimately provide a framework for deciding between a moment of creation and an eternal universe.<sup>1</sup> On the one hand various toy models for cosmological singularities in string theory have been developed, and considerable effort has been devoted to studying them, but a complete understanding is still lacking [5]. On the other hand several string-inspired models for eternal cosmologies have been proposed, most notably the ekpyrotic [6] and pre-big bang [7] scenarios, but it is not clear to what extent these proposals capture the generic (or even allowed) behavior of string theory.

In view of this situation it is worthwhile developing additional scenarios for eternal cosmologies in string theory. In this paper we consider a simple class of higher-derivative modifications to the effective action for gravity. These modifications have the effect of bounding the expansion rate and limiting dilaton gradients, thereby avoiding singularities at any finite time. In the absence of matter the universe would approach a de Sitter phase at early times. But when coupled to a gas of string winding and momentum modes the scale factors can oscillate or bounce as functions of time. By introducing a dilaton potential the dilaton can be made to oscillate or bounce as well. Our work has several motivations.

## *Bouncing and cyclic cosmologies*

Eternal cosmologies in which the scale factors bounce or oscillate as functions of time have been extensively studied, and within field theory a variety of mechanisms for realizing this type of behavior have been developed [2]. Our work provides a simple string-inspired mechanism for obtaining bouncing and cyclic cosmologies. For other studies in this direction see [8, 9, 10].

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<sup>1</sup>Eternal inflation, despite its name, is not past-eternal and cannot by itself address this issue [3, 4].

### *Pre-big bang scenario*

In the pre-big bang scenario the universe is assumed to begin from a cold flat weakly-coupled initial state. The dilaton rolls towards strong coupling and bounces, and the universe emerges in an expanding FRW phase [7]. However at the level of the two-derivative effective action the two branches of pre-big bang cosmology cannot be smoothly connected [11]. There are various ways around this, reviewed in section 8 of [7]. Our work leads to a particularly simple realization of the pre-big bang scenario, in a manner similar to the proposal [12].

### *Horizon problem*

One of the main puzzles of conventional FRW cosmology is the horizon problem: how did causally-disconnected regions of the universe come to be in thermal equilibrium? Inflation explains this by postulating a rapid growth of the scale factor at early times. But an alternative way to address the horizon problem is to postulate a loitering phase in which the scale factor is roughly constant (that is, the Hubble length diverges). If the loitering phase lasts long enough the universe has time to come to thermal equilibrium. The models we discuss can have long loitering phases, a phenomenon observed in a similar context in [13].

### *Brandenberger-Vafa mechanism*

As a zeroth-order goal, one might hope that string cosmology could account for the three large spatial dimensions we observe. An intriguing dynamical mechanism for obtaining three large dimensions was proposed by Brandenberger and Vafa [14]; see [15] for a review. They imagine the universe began at the Hagedorn temperature, with all spatial dimensions compactified on a string-scale torus. A gas of winding modes keeps the universe from expanding, while a gas of momentum modes keeps it from shrinking, until a thermal fluctuation happens to make some number of dimensions expand. If three or fewer dimensions expand the winding strings should generically intersect, and if they happen to annihilate there will be no obstacle to those dimensions decompactifying. But four or more dimensions should be prevented from decompactifying, since the winding strings will generically not intersect. Although appealing, at the level of the two-derivative effective action this scenario has a problem: the universe has a singularity a finite time in the past. Moreover the dilaton rolls monotonically towards weak coupling, so if one waits too long strings are unlikely to annihilate even if they do happen to intersect. This means there is only a small window of time for the necessary thermal fluctuations to take place, and as a result three large dimensions are not statistically favored [16]. The eternal cosmologies we discuss would seem to provide a natural setting for realizing the

Brandenberger-Vafa mechanism. Indeed this was our original motivation for analyzing these models. We will find that, due to possibility of long loitering phases, our models do not preferentially decompactify three dimensions.

Several of the results we will obtain have antecedents in the literature, in particular in the works [12, 13], although to our knowledge the phenomena we will discuss have never appeared in combination before. An outline of this paper is as follows. In section 2 we introduce a modified action for Einstein-dilaton gravity which has the effect of bounding the expansion rate and dilaton gradient. In section 3 we introduce matter degrees of freedom and give a preliminary discussion of their thermodynamics. In section 4 we consider the coupled gravity – matter system and show how bouncing and cyclic cosmologies result. In section 5 we study string interactions and entropy production in these cosmologies. In section 6 we show that entropy production drives an eventual transition to a conventional radiation-dominated cosmology. Section 7 studies the extent to which the Brandenberger-Vafa mechanism is operative in these models. We conclude in section 8. In appendix A we discuss energy conditions in dilaton gravity and in appendix B we study the fine-tuning of initial conditions required to exit the Hagedorn phase if one uses a two-derivative effective action.

## 2 A modified action

We consider type II string theory compactified on a torus with metric

$$ds^2 = -dt^2 + \alpha' \sum_{i=1}^d e^{2\lambda_i(t)} dx_i^2 \quad x_i \approx x_i + 2\pi. \quad (1)$$

From now on we set  $\alpha' = 1$ . Although we have in mind that all spatial dimensions are compactified, we will allow  $d$  to vary to study the dimension dependence of our results. At the two-derivative level the string-frame effective action for homogeneous fields takes the form

$$S_0 = \int dt \left[ 4\pi^2 e^{-\varphi} \left( \sum_i \dot{\lambda}_i^2 - \dot{\varphi}^2 \right) + L_{\text{matter}} \right]. \quad (2)$$

The action includes standard kinetic terms for the radii and dilaton; we're working in terms of a shifted dilaton  $\varphi$ , related to the usual dilaton  $\phi$  by [17]

$$\varphi = 2\phi - \sum_i \lambda_i. \quad (3)$$

$L_{\text{matter}}$  is the effective Lagrangian for matter degrees of freedom. In thermal equilibrium we'll identify  $L_{\text{matter}} = -F$  with the negative of the matter free energy. Besides the equations of motion which follow from this action we have the Hamiltonian constraint that the total energy in the universe vanishes.

$$\dot{\varphi}^2 - \sum_i \dot{\lambda}_i^2 = \frac{1}{4\pi^2} E e^\varphi \quad (4)$$

Here  $E$  is the energy in matter. These equations are invariant under T-duality, which acts according to

$$\begin{aligned} \lambda_i &\rightarrow -\lambda_i && \text{for some } i \\ \varphi, L_{\text{matter}}, E &&& \text{invariant} \end{aligned}$$

Provided  $E$  satisfies certain energy conditions the action  $S_0$  leads to cosmologies that have initial singularities: at a finite proper time in the past the shifted dilaton diverges and the  $\lambda_i \rightarrow \pm\infty$ . These singularities have mostly been studied in the context of pre-big bang cosmology, see for example [18] and [19]. But it is tempting to speculate that stringy effects ( $\alpha'$  corrections to the effective action) will lead to non-singular cosmologies.<sup>2</sup> As a model which captures this sort of behavior, we introduce the following modified action for the metric and dilaton.

$$S = \int dt \left[ 8\pi^2 e^{-\varphi} \left( \sqrt{1 - \dot{\varphi}^2} - \sqrt{1 - \sum_i \dot{\lambda}_i^2} \right) + L_{\text{matter}} \right] \quad (5)$$

Again in equilibrium we'll identify  $L_{\text{matter}} = -F$  with the negative of the matter free energy.

There are several motivations for writing down this action. As a simple way to think about it, note that in the action (2) both  $\varphi$  and  $\lambda_i$  appear as non-relativistic particles of mass  $8\pi^2 e^{-\varphi}$  (although  $\varphi$  has a wrong-sign kinetic energy). In going from (2) to (5) we have promoted  $\varphi$  and  $\lambda_i$  to become relativistic particles of the same mass. This clearly bounds their velocities,

$$\dot{\varphi}^2 < 1 \quad \sum_i \dot{\lambda}_i^2 < 1$$

which has the desired effect of ruling out singularities at any finite proper time. In this sense the action we have written down incorporates a “limiting curvature hypothesis”

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<sup>2</sup>For a study of  $\alpha'$  corrections in string gas models see [20].

in a manner similar to [12, 21]. It's also amusing to note the resemblance of  $S$  to the DBI action for open strings, which is related by T-duality to the action for a relativistic particle [22]. Finally we note that with the conventional two-derivative action (2), some fine-tuning of initial conditions is required to exit an initial Hagedorn phase. We discuss this in appendix B. With the modified action (5), this difficulty is avoided.

For simplicity we specialize to a square torus with all  $\lambda_i = \lambda$ . Then the equations of motion which follow from  $S$  are

$$\begin{aligned}\dot{\gamma}_\varphi &= \dot{\varphi}(\gamma_\varphi - \gamma_\lambda^{-1}) + \frac{1}{8\pi^2} \dot{\varphi} e^\varphi P_\varphi \\ \dot{\gamma}_\lambda &= \dot{\varphi}(\gamma_\lambda - \gamma_\lambda^{-1}) + \frac{1}{8\pi^2} d\dot{\lambda} e^\varphi P_\lambda\end{aligned}\tag{6}$$

Here  $P_\varphi = \frac{\partial F}{\partial \varphi}$  is the force on the dilaton and  $P_\lambda = -\frac{1}{d} \frac{\partial F}{\partial \lambda}$  is the pressure (or more accurately, the pressure times the volume of the torus).<sup>3</sup> We have defined the relativistic factors

$$\gamma_\varphi = \frac{1}{\sqrt{1 - \dot{\varphi}^2}} \quad \gamma_\lambda = \frac{1}{\sqrt{1 - d\dot{\lambda}^2}}.\tag{7}$$

The Hamiltonian constraint (Friedmann equation) is

$$\gamma_\varphi - \gamma_\lambda = \frac{1}{8\pi^2} E e^\varphi\tag{8}$$

where  $E$  is the matter energy. Note that the positive energy region is  $|\dot{\varphi}| \geq \sqrt{d}|\dot{\lambda}|$ , just as in lowest order dilaton gravity.

To get oriented, consider a simple equation of state  $P_\lambda = wE$ ,  $w$  constant, with  $P_\varphi = 0$ . Of particular interest are the cases  $w = 0$ ,  $w = 1/d$  and  $w = -1/d$  which correspond to a Hagedorn era, a radiation dominated era and a winding mode dominated era respectively. One can get an idea of how the system evolves by writing equations for  $\ddot{\varphi}(\dot{\varphi}, \dot{\lambda})$  and  $\ddot{\lambda}(\dot{\varphi}, \dot{\lambda})$  and studying the phase space flow. Using the above equation of state, and substituting the Hamiltonian constraint in the equation for  $\lambda$  we can write

$$\begin{aligned}\ddot{\varphi} &= (1 - \dot{\varphi}^2)(1 - \gamma_\lambda^{-1} \gamma_\varphi^{-1}) \\ \ddot{\lambda} &= (1 - d\dot{\lambda}^2)(\dot{\varphi} \dot{\lambda} - w(1 - \gamma_\lambda^{-1} \gamma_\varphi))\end{aligned}\tag{9}$$

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<sup>3</sup>These are derivatives at fixed temperature. Entropy will be conserved, until we consider out-of-equilibrium processes in section 5, so it's perhaps more appropriate to write  $P_\lambda = -\frac{1}{d} \left( \frac{\partial E}{\partial \lambda} \right)_S$  as a derivative at fixed entropy.

It is easy to see that these equations have fixed points at the constant curvature, linear dilaton solutions  $(\dot{\varphi}, \dot{\lambda}) = (\pm 1, \pm 1/\sqrt{d})$ . These can be smoothly connected to the trivial fixed point  $(\dot{\varphi}, \dot{\lambda}) = (0, 0)$  in the sense that no singularity stands between them. This is an attractive feature that  $\alpha'$  corrections to the low energy effective action are conjectured to have, perhaps to all orders in  $\alpha'$  [23]. It is particularly relevant to pre-big bang models. The phase space flows and some trajectories for  $d = 3$  are shown in figure 1.<sup>4</sup> In such a smooth and “connected” phase space, the system can move around the phase space towards the attractors without encountering singularities, *independently of initial conditions*. This feature is hard to obtain with generic  $\alpha'$  corrections to dilaton gravity and is crucial for the cyclic and bouncing solutions we will study below. For example, with the conventional two-derivative action for dilaton gravity one could at most hope for a single bounce before encountering a singularity. In essence, with the new action, we have replaced these singularities with the constant velocity fixed points.

For general matter content there is a simple way to see how the modified equations of motion capture the desired behavior. Assuming that  $E$  is positive the Friedmann equation requires  $\gamma_\varphi > \gamma_\lambda \geq 1$  so  $\dot{\varphi}$  can never vanish. Orienting time so that  $\dot{\varphi} < 0$ , the dilaton rolls monotonically from strong to weak coupling. Since  $\varphi$  and  $\lambda$  behave like relativistic particles of mass  $\sim e^{-\varphi}$ , at early times they are massless and move at the speed of light:

$$\dot{\varphi} \rightarrow -1 \quad \dot{\lambda} \rightarrow \pm 1/\sqrt{d} \quad \text{as } t \rightarrow -\infty. \quad (10)$$

Thus at early times the scale factors grow exponentially and the metric (1) approaches de Sitter space in planar coordinates, with the spatial coordinates periodically identified to make a torus. This early-time de Sitter phase is what replaces the big bang singularity in these models.<sup>5</sup> This is very reminiscent of the behavior obtained in [21]. It is also similar to pre-big bang models where the kinetic energy of the dilaton dominates and drives inflation. One might worry about the fact that the coupling diverges at early times; as we will see we can cure this behavior by introducing a potential for the dilaton which violates positivity of  $E$ .

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<sup>4</sup>For  $w \neq 0$  the equations of motion are singular when  $|\dot{\varphi}| = 1$ . This is not problematic because trajectories never quite reach points where  $|\dot{\varphi}| = 1$ . Instead the  $\gamma_\lambda^{-1}\gamma_\varphi$  term in the equation for  $\ddot{\lambda}$  eventually pushes the trajectories towards the line  $\dot{\varphi} = \pm\sqrt{d}\dot{\lambda}$  (depending on the sign of  $w$ ) where  $\gamma_\lambda^{-1}\gamma_\varphi \rightarrow 1$ .

<sup>5</sup>The dilaton diverges at  $t = -\infty$ , so strictly speaking we have not eliminated the singularity, just moved it infinitely far into the past. As we will see even this can be cured by adding a potential for the dilaton.

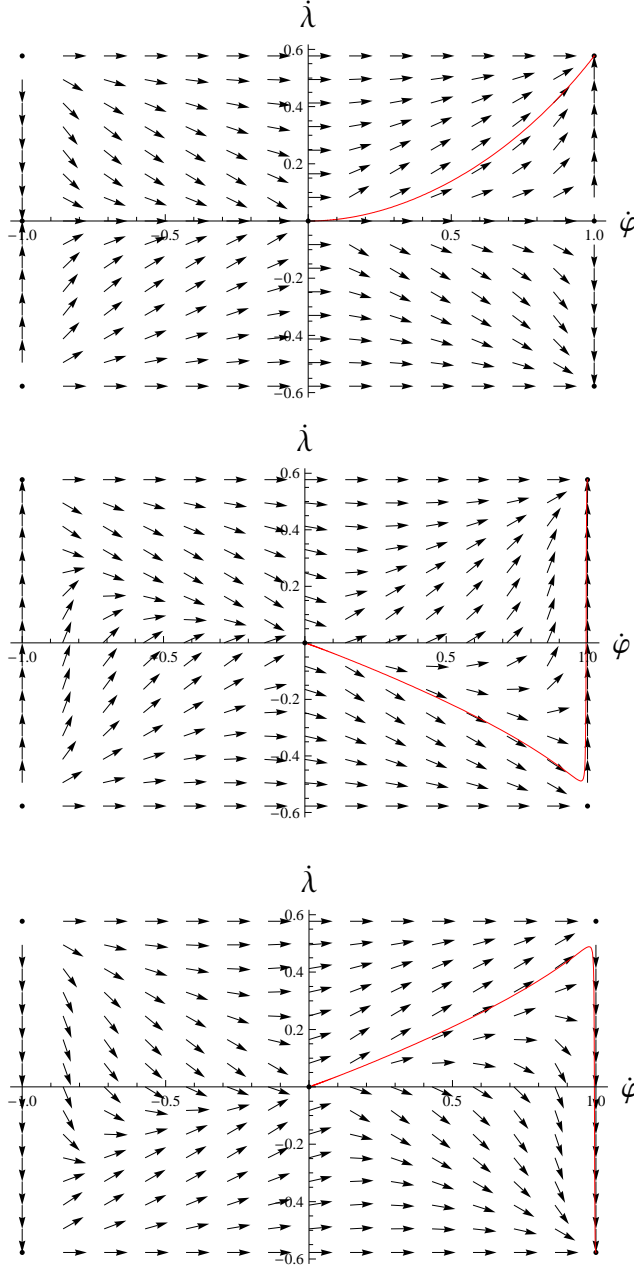


Figure 1: Phase space flows for  $w = 0$  (top),  $w = 1/d$  (middle),  $w = -1/d$  (bottom). The five fixed points  $(\dot{\varphi}, \dot{\lambda}) = (\pm 1, \pm 1/\sqrt{d})$  and  $(\dot{\varphi}, \dot{\lambda}) = (0, 0)$  are connected smoothly. Some typical trajectories are also shown. For  $w = 1/d$  and  $w = -1/d$  they represent bounces of the scale factor due to KK and winding modes respectively.



### 3 A first pass at thermodynamics

To proceed further we need to specify the matter content of the universe. We will be fairly conservative at this stage, since we've already modified the dilaton gravity action to eliminate singularities. Readers familiar with these standard results may skip ahead to the next section.

We take matter to consist of the following ingredients.

1. There may be a gas of string winding modes, characterized by winding numbers  $W_i$  that count the number of strings wound with positive orientation around the  $i^{th}$  dimension of the torus.<sup>6</sup> For simplicity we set all  $W_i = W$ . Then the energy in winding modes is

$$E_W = 2dWe^\lambda.$$

2. Likewise there may be a gas of Kaluza-Klein momentum modes, characterized by positively-oriented momentum numbers  $K_i$ . With all  $K_i = K$ , the energy in Kaluza-Klein modes is

$$E_K = 2dKe^{-\lambda}.$$

3. We allow for a gas of string oscillator modes which we will model as pressureless dust with energy  $E_{\text{dust}}$ .

To be precise,  $W$  and  $K$  refer to the winding and momentum numbers in the first  $d$  dimensions. Thus we take  $E_{\text{dust}}$  to represent the energy, not only in string oscillators, but also in winding and momentum modes in the remaining  $9 - d$  dimensions. These modes can be modeled as dust since they do not contribute to the pressure in the first  $d$  dimensions. As the remaining component of the energy budget, we may introduce a potential for the dilaton  $V(\varphi)$ . The total energy is then the sum

$$E = E_W + E_K + E_{\text{dust}} + V. \quad (11)$$

Treating the system adiabatically the “pressures” are

$$P_\varphi = \frac{\partial E}{\partial \varphi} = \frac{\partial V}{\partial \varphi} \quad (12)$$

$$P_\lambda = -\frac{1}{d} \frac{\partial E}{\partial \lambda} = 2Ke^{-\lambda} - 2We^\lambda \quad (13)$$

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<sup>6</sup>Since we work in a compact space there must be an equal number of strings wound with the opposite orientation.

We will assume that the dilaton potential is independent of temperature. However the other components of the energy budget behave thermodynamically. To make the distinction, we refer to  $E_s \equiv E_W + E_K + E_{\text{dust}}$  as the energy in the *string gas*, the thermodynamical component of the total energy. The following phases will be of interest to us [24].

#### *Hagedorn phase*

In the Hagedorn phase we assume that all matter degrees of freedom are in thermal equilibrium at the type II Hagedorn temperature  $T_H = 1/(\sqrt{8}\pi)$ . Hagedorn thermodynamics has been studied extensively [14, 25, 26, 27]. The free energy of the string gas vanishes, so  $P_\lambda = 0$  and the energy  $E_s$  is conserved; since  $E_s = T_H S$  the entropy is also conserved. In equilibrium the winding and momentum numbers are<sup>7</sup>

$$\langle W \rangle = \frac{\sqrt{E_s}}{12\sqrt{\pi}} e^{-\lambda} \quad \langle K \rangle = \frac{\sqrt{E_s}}{12\sqrt{\pi}} e^{\lambda} \quad (14)$$

As expected these values make the pressure in (13) vanish.

#### *Radiation phase*

The radiation phase describes the equilibrium situation at temperatures  $T < T_H$ . The universe is dominated by a gas of massless string modes with energy

$$E_s = c_d V_d T^{d+1}. \quad (15)$$

Here  $c_d$  is a constant appropriate to a gas of 128 massless Bose and 128 massless Fermi degrees of freedom,

$$c_d = 128 \frac{2d! \zeta(d+1)}{(4\pi)^{d/2} \Gamma(d/2)} (2 - 2^{-d}). \quad (16)$$

Also  $V_d = (2\pi)^d e^{d|\lambda|}$  is the T-duality invariant “volume” of the torus. This definition takes into account the fact that the energy could be stored in either momentum or winding modes depending on the size of the torus. We have the standard thermodynamic results

$$F = E_s - TS = -\frac{1}{d} c_d V_d T^{d+1} \quad (17)$$

$$P_\lambda = \text{sign}(\lambda) E_s / d \quad (18)$$

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<sup>7</sup>These values follow from the distributions in [26] with the assumption that the energy is equally partitioned among all compact dimensions [16].

leading as usual to a conserved entropy. We also have the equilibrium values

$$\begin{aligned} \lambda > 0 : \quad & \langle W \rangle = 0 & \langle K \rangle = \frac{1}{2} P_\lambda e^\lambda \\ \lambda < 0 : \quad & \langle W \rangle = -\frac{1}{2} P_\lambda e^{-\lambda} & \langle K \rangle = 0 \end{aligned} \tag{19}$$

These follow from requiring that the pressure (13) takes on the correct value.

#### *Frozen phase*

Finally as an alternative to an equilibrium radiation phase we consider a frozen phase in which the interactions between strings are turned off. The momentum and winding numbers are conserved, so  $K$  and  $W$  are frozen at the values which they have on Hagedorn exit. Any remaining energy in the universe goes into dust. As we discuss in section 5, in this phase matter entropy is conserved. In section 5 we will go beyond this approximation and study entropy production due to interactions between winding and momentum modes. But please note that we will refer to an out-of-equilibrium string gas as being in a radiation phase if the equilibrium temperature would be below Hagedorn.

## 4 Bouncing and cyclic cosmologies

In this section we study what happens when we couple the modified dilaton-gravity action of section 2 to a gas of string winding and momentum modes. For simplicity we will model the string gas using just the Hagedorn and frozen phases described in section 3. This is not very realistic, but it will serve to illustrate the way a string gas changes the dynamics. We will give a more realistic treatment in section 6.

First let's see what happens for vanishing dilaton potential. Whether we're in a Hagedorn or frozen phase the matter energy is positive, so as shown above (10) the dilaton will roll monotonically from strong to weak coupling. Suppose we're at strong coupling, and let's assume we're in equilibrium in the Hagedorn phase with  $\dot{\lambda} > 0$ . Since we're at strong coupling the fields  $\varphi$  and  $\lambda$  behave like massless particles.<sup>8</sup> Moreover there's no force on these particles: with no dilaton potential  $P_\varphi = 0$ , and in the Hagedorn phase  $P_\lambda = 0$ . So the particles move at nearly the speed of light,

$$\dot{\varphi} \approx -1 \quad \dot{\lambda} \approx 1/\sqrt{d}. \tag{20}$$

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<sup>8</sup>We will be more precise about this in (22) below.

But this behavior cannot persist indefinitely. As the universe expands eventually it will cool below the Hagedorn temperature. To see when this happens we compute the energy  $E$  in matter using the Friedmann equation (8). Then we compute the equilibrium radiation temperature  $T_{\text{rad}}$  using (15). If  $T_{\text{rad}} < T_H$  the universe is no longer in the Hagedorn phase. But rather than go to an equilibrium radiation phase, we assume the universe makes a transition to a frozen phase in which the momentum and winding numbers  $K$  and  $W$  are conserved, equal to whatever values they had on Hagedorn exit.

In the frozen phase the pressure does not vanish. Instead there is an effective potential for the scale factor,

$$V(\lambda) = E_W + E_K = 2dW e^\lambda + 2dK e^{-\lambda}. \quad (21)$$

At some point  $\lambda$  bounces off this potential. The universe shrinks and eventually re-enters a Hagedorn phase. It subsequently emerges from this new Hagedorn phase and undergoes a T-dual bounce, driven by momentum modes, at  $\lambda < 0$ . The whole cycle repeats, resulting in an oscillating scale factor. However the oscillations cannot persist indefinitely. When the dilaton reaches weak coupling the  $\phi$  and  $\lambda$  particles become very massive and come to rest, putting an end to the oscillations. This can be seen in a numerical solution in figure 2. Note that at strong coupling the oscillations have constant amplitude. This is a consequence of neglecting interactions, which implies no entropy production in the frozen phase: the system always re-enters the Hagedorn phase with the same values of  $\lambda$  and  $E$ , which in the Hagedorn phase corresponds to the system having the same entropy. We will relax this approximation in section 5.

What we need for a cyclic scale factor is not strong coupling, necessarily, but rather a large amount of energy stored in the dilaton. This can be seen from the Friedmann equation

$$\gamma_\lambda = \frac{1}{8\pi^2} (E_\varphi - E_s) e^\varphi \quad (22)$$

where  $E_\varphi = 8\pi^2 \gamma_\varphi e^{-\varphi} - V(\varphi)$  is the (negative of) the total energy stored in the dilaton, and  $E_s$  is the energy in the string gas. As long as  $E_\varphi$  is large enough the scale factor is relativistic and can undergo bounces in a suitable potential.

It is useful to note that when  $\gamma_\lambda \gg 1$  the equations of motion (6) imply  $\dot{\gamma}_\varphi \approx \dot{\varphi} \gamma_\varphi + \frac{1}{8\pi^2} \dot{\varphi} e^\varphi \frac{\partial V(\varphi)}{\partial \varphi}$ , so  $\frac{d}{dt} (8\pi^2 \gamma_\varphi e^{-\varphi} - V(\varphi)) \approx 0$  and  $E_\varphi$  is conserved. In the Hagedorn phase  $E_s$  is conserved as well, so (22) gives a clear picture of the dynamics: in the frozen phase, as  $\lambda$  grows the winding modes (or KK modes in the dual picture) absorb

energy and increase  $E_s$  until  $\gamma_\lambda$  drops to 1 and the universe bounces. A plot of  $E_s$  is shown in figure 3.

So far we have discussed solutions in which the dilaton evolves monotonically. However the dilaton need not run to infinite coupling in the far past. A past state for the universe could be one where the expansion rate is arbitrarily small and the string coupling is arbitrarily weak. Provided  $\dot{\varphi} \gtrsim 0$  at early times, a simple dilaton potential of the form  $V(\varphi) = Ae^\varphi$  with  $A < 0$  can generate a bounce for  $\varphi$  and turn it back toward weak coupling at late times. This is the basic idea of the pre-big bang scenario. A numerical solution is shown in figure 4.

As a further example, an upside down potential of the form  $V(\varphi) = Ae^\varphi + Be^{-\varphi}$ , with  $A$  and  $B$  negative, can restrict the dilaton to vary within a finite range. The dilaton will undergo bounces, just like the scale factor, with  $E_\varphi$  converting between large negative kinetic energy and large negative potential energy. A typical numerical solution is shown in figure 5.<sup>9</sup>

So far we have discussed bouncing and cyclic behavior using the string frame metric. Since we have in mind coupling to stringy matter this is the physically relevant frame to use. However one might be interested in the behavior of the Einstein frame metric, with scale factor

$$\lambda_E = -\frac{1}{d-1}(\varphi + \lambda).$$

If the matter energy is positive, implying that the dilaton evolves monotonically, then the Einstein frame scale factor will evolve monotonically as well: the Friedmann equation (8) requires  $\dot{\varphi}^2 > d\dot{\lambda}^2$ . However the models with a bouncing dilaton lead to a bouncing scale factor in Einstein frame. Generically each bounce of the dilaton will correspond to a bounce of  $\lambda_E$ .

One might worry that we have introduced dilaton potentials which are unbounded below. However note that our solutions only explore a limited range of  $\varphi$ , and one could easily imagine obtaining the same behavior from a stable potential, just by modifying  $V(\phi)$  outside the range of variation of the dilaton. One might also worry that bouncing and cyclic cosmologies require violation of certain energy conditions. We address this in appendix A.

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<sup>9</sup>The initial conditions in figures 2, 4 and 5 are chosen such that when  $\gamma_\lambda \gg 1$ ,  $E_\varphi$  has the same value in all three examples. The Hagedorn phase  $E_s$  is also chosen to be the same (smaller than  $E_\varphi$ ). These two energies determine the amplitude of the cycles, as we will see in more detail in section 6, so the maximum value of  $\lambda$  is the same in all three figures.

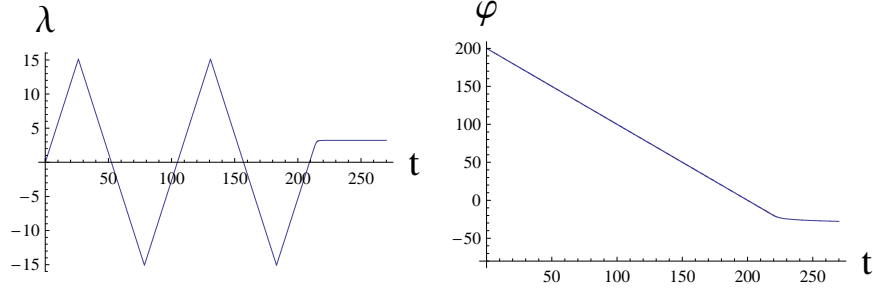


Figure 2: Numerical solution with Hagedorn and frozen phases and no potential for the dilaton. The oscillations have constant amplitude as there is no entropy production. The oscillations stop when the universe reaches weak coupling. We use  $d = 3$ , as in all graphs that follow.

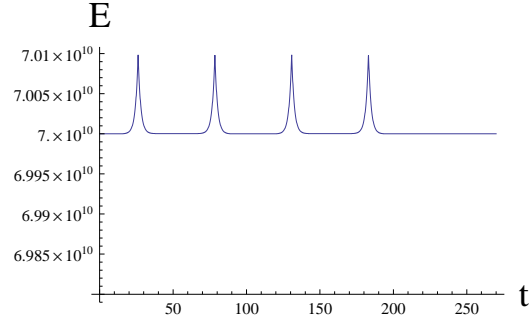


Figure 3: A plot of the energy in the string gas for Fig. 2. The energy is constant during the Hagedorn phases. During a frozen phase it increases until the scale factor bounces. It then decreases and the system re-enters the Hagedorn phase.

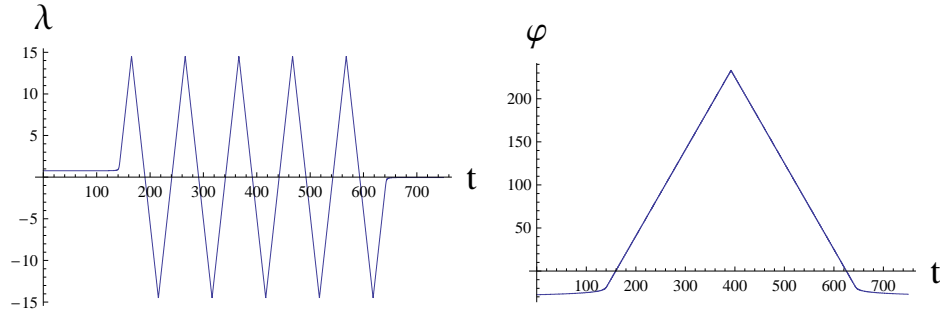


Figure 4: Same as Fig. 2, but with a dilaton potential of the form  $Ae^\varphi$  that yields a single bounce for the dilaton.

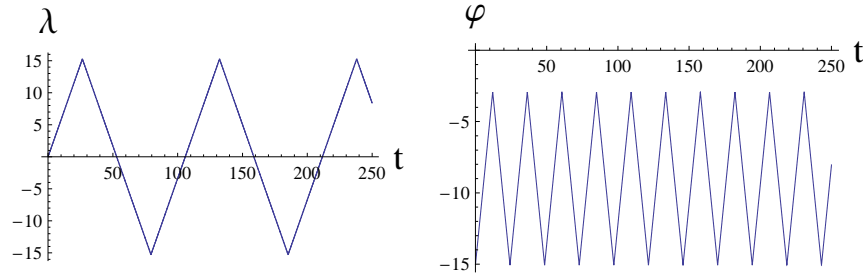


Figure 5: A potential of the form  $Ae^\varphi + Be^{-\varphi}$  can confine the dilaton at weak coupling.

## 5 Interactions and entropy production

In this section we study the effect of interactions on an out-of-equilibrium string gas. We will continue to assume that thermal equilibrium holds during the Hagedorn phase, but we will allow the momentum and winding modes to go out of equilibrium in the radiation phase, where the temperature is below Hagedorn. We first take a macroscopic thermodynamic perspective and discuss entropy production, then present Boltzmann equations for the winding and momentum numbers. For simplicity in this section we will neglect the possibility of having a dilaton potential.

Our goal is to understand how the momentum and winding numbers  $K$  and  $W$  evolve towards their equilibrium values. One constraint comes from energy conservation. The equations of motion (6) along with the Hamiltonian constraint (8) imply

that

$$\dot{E} = -dP_\lambda \dot{\lambda}. \quad (23)$$

Here the dot indicates a time derivative and  $d$  is the number of dimensions, not a differential. Breaking up the matter energy as in (11), namely  $E = E_W + E_K + E_{\text{dust}}$ , and likewise breaking up the pressure, the energy conservation equation (23) becomes

$$\dot{E}_W + \dot{E}_K + \dot{E}_{\text{dust}} = -d(P_W + P_K)\dot{\lambda}. \quad (24)$$

For the individual species we have

$$\begin{aligned} \dot{E}_W &= \frac{d}{dt}(2dWe^\lambda) = 2dWe^\lambda \dot{\lambda} + 2d\dot{W}e^\lambda \\ &= -dP_W \dot{\lambda} + 2d\dot{W}e^\lambda \end{aligned} \quad (25)$$

and

$$\begin{aligned} \dot{E}_K &= \frac{d}{dt}(2dKe^{-\lambda}) = -2dKe^{-\lambda} \dot{\lambda} + 2d\dot{K}e^{-\lambda} \\ &= -dP_K \dot{\lambda} + 2d\dot{K}e^{-\lambda} \end{aligned} \quad (26)$$

Combining (24), (25) and (26), we must have

$$\dot{E}_{\text{dust}} + 2d(\dot{W}e^\lambda + \dot{K}e^{-\lambda}) = 0 \quad (27)$$

in order for energy to be conserved.

Another constraint comes from the second law of thermodynamics. To illustrate what's required let's temporarily model the universe as filled with two fluids at different temperatures. One fluid consists of pressureless dust and winding modes and is held at the Hagedorn temperature  $T_H$ , the other consists of radiation (i.e. momentum modes) held at temperature  $T_K$ . The two fluids are out of equilibrium when  $T_K < T_H$  which is what we expect to occur when we exit the Hagedorn phase. The resulting entropy production rate is

$$\begin{aligned} \dot{S} &= \dot{S}_K + \dot{S}_W + \dot{S}_{\text{dust}} \\ &= \frac{\dot{E}_K + dP_K \dot{\lambda}}{T_K} + \frac{\dot{E}_W + dP_W \dot{\lambda}}{T_H} + \frac{\dot{E}_{\text{dust}}}{T_H} \\ &= \frac{2d\dot{K}e^{-\lambda}}{T_K} + \frac{\dot{E}_{\text{dust}} + 2d\dot{W}e^\lambda}{T_H} \end{aligned} \quad (28)$$

and using (27)

$$\dot{S} = 2d\dot{K}e^{-\lambda} \left( \frac{1}{T_K} - \frac{1}{T_H} \right). \quad (29)$$



Provided  $\dot{K}$  is positive (radiation is produced) whenever  $T_K < T_H$ , we will have  $\dot{S} > 0$  consistent with the second law of thermodynamics. So in the frozen phase of section 3, where  $K$  was conserved, there was no entropy production. But any sensible evolution equation for  $K$  and  $W$  will lead to an increase in entropy.

We now present such an evolution equation for the winding number  $W$ . At weak string coupling the appropriate Boltzmann equation was derived in [16], based on the cross section for winding – anti-winding annihilation obtained in [28].

$$\dot{W} = -\frac{e^{2\lambda+\varphi}}{\pi} (W^2 - \langle W \rangle^2) \quad (30)$$

Here  $\langle \cdot \rangle$  denotes a thermal expectation value, given in (19) for temperatures below Hagedorn. This expression makes intuitive sense: the factor  $e^{2\lambda}$  captures the fact that longer strings are more likely to annihilate, while the factor  $e^\varphi = g_s^2/V$  takes into account both enhancement by the string coupling  $g_s$  and suppression by the volume of the torus  $V$ . The result (30) is reliable at weak coupling, but we will often be interested in behavior at strong coupling. At strong coupling we adopt the following modified Boltzmann equation.

$$\dot{W} = -\frac{e^{2\lambda-d|\lambda|}}{\pi} (W^2 - \langle W \rangle^2) \quad (31)$$

This equation can be obtained from the previous weak-coupling Boltzmann equation (30) by making the replacement  $\varphi \rightarrow -d|\lambda|$ , that is, by dropping the unshifted dilaton from the cross section but keeping the dependence on the T-duality-invariant “volume”  $\exp(-d|\lambda|)$ . This can be thought of purely phenomenologically, as describing winding strings (such as cosmic strings) whose interactions do not depend on the unshifted dilaton. It can also be regarded as describing fundamental strings, but with a potential for the dilaton that fixes the unshifted dilaton to  $\phi \approx 0$ . For momentum modes at strong coupling we use the T-dual equation

$$\dot{K} = -\frac{e^{-2\lambda-d|\lambda|}}{\pi} (K^2 - \langle K \rangle^2) . \quad (32)$$

## 6 Shrinking cycles and exit

We now study how entropy production in an out-of-equilibrium string gas affects the cyclic cosmologies of section 4. For simplicity we set the dilaton potential to zero.

Recall that in section 4 we neglected interactions during the frozen phase; the momentum and winding numbers were taken to be conserved. This led to a constant entropy and oscillations of fixed amplitude. Taking interactions into account we will see that the resulting entropy production leads to oscillations of decreasing amplitude. Oscillating models often exhibit this sort of behaviour, but the details depend on the mechanism that drives the bounce [29]. For example in a recent bouncing cosmology, in which an equilibrium Hagedorn era was also used, the oscillations grew with time [30]. But in this model there was no dilaton and the bounce was driven by positive spatial curvature and negative Casimir energy.

In our models eventually so much entropy is produced that it is no longer thermodynamically possible for the universe to re-enter the Hagedorn phase. At this point the universe transitions to a loitering phase in which the scale factors are roughly constant, oscillating about a minimum in their potential. Eventually the loitering phase also ends and the universe transitions to a standard radiation-dominated cosmology.

## 6.1 Shrinking cycles

The dynamics are largely governed by the energy stored in the dilaton. We are neglecting any dilaton potential, so as noted in section 4 the (negative of) the dilaton kinetic energy

$$E_{\max} \equiv 8\pi^2 e^{-\varphi} \gamma_{\varphi} \quad (33)$$

is essentially constant. We have denoted this  $E_{\max}$  because it's equal to the maximum matter energy during a cycle. To see this recall that the Friedmann equation (8) states that the energy in matter is

$$E = 8\pi^2 e^{-\varphi} (\gamma_{\varphi} - \gamma_{\lambda}) . \quad (34)$$

At a bounce we have  $\gamma_{\varphi} \gg \gamma_{\lambda} = 1$  and therefore  $E \approx E_{\max}$ .

During the radiation phase of the  $n^{\text{th}}$  cycle the energy in matter starts at  $E_n$ , the (conserved) matter energy during the Hagedorn phase of the  $n^{\text{th}}$  cycle. It increases to  $E_{\max}$  as the wound strings are stretched.<sup>10</sup> After the bounce the matter energy decreases down to the value  $E_{n+1}$  associated with the next Hagedorn phase. These

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<sup>10</sup>For simplicity we discuss bounces at large radius. At small radius T-duality would exchange momentum and winding.

Hagedorn phases serve as reference equilibrium points in phase space where the entropy is well defined, given by  $S_n = E_n/T_H$ . Since entropy is produced during the radiation phase,  $S_{n+1} > S_n$  as we saw above, and since we return to the same (equilibrium) temperature  $T_H$  when re-entering the Hagedorn phase, the matter energy must increase during each radiation phase as well,  $E_{n+1} > E_n$ . This means the radius at which we exit the Hagedorn phase also increases with each cycle. To see this recall that the condition for exit is that the equilibrium radiation temperature drops below Hagedorn.<sup>11</sup> From (15) this means that at Hagedorn exit

$$E_n = c_d (2\pi)^d e^{d\lambda_n} T_H^{d+1}. \quad (35)$$

Since  $E_n$  increases with each cycle, so does the scale factor at exit  $e^{\lambda_n}$ .

We can also estimate the maximum scale factor reached during each cycle  $e^{\lambda_n^{\max}}$ . From the moment of Hagedorn exit to the bounce, matter energy increases by an amount

$$E_{\max} - E_n = -d \int_{\lambda_n}^{\lambda_n^{\max}} d\lambda P_\lambda \approx 2d \int_{\lambda_n}^{\lambda_n^{\max}} d\lambda (W_n e^\lambda - K_n e^{-\lambda}) \quad (36)$$

where we've assumed interactions are weak so the values at Hagedorn exit

$$W_n = \frac{\sqrt{E_n}}{12\sqrt{\pi}} e^{-\lambda_n} \quad K_n = \frac{\sqrt{E_n}}{12\sqrt{\pi}} e^{\lambda_n} \quad (37)$$

are roughly conserved. This leads to

$$e^{\lambda_n^{\max}} \sim E_n^{1/d} \left( \alpha(E_n) + \sqrt{\alpha(E_n)^2 - 1} \right) \quad (38)$$

where

$$\alpha(E_n) = \frac{3}{d} \sqrt{\frac{\pi}{E_n}} (E_{\max} - E_n) + 1.$$

$\lambda_n^{\max}$  is a decreasing function of  $E_n$ , so the maximum radius shrinks with each cycle.

The features we have discussed can be seen in figure 6, which shows a numerical solution to the combined gravitational equations of motion (6), (8) and the strong-coupling Boltzmann equations (31), (32). The matter energy  $E$  has plateaus which

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<sup>11</sup>Note that no *real* temperature is dropping here since during the Hagedorn phase the temperature is constant at  $T_H$ . By equilibrium radiation temperature we mean the temperature that radiation alone would have in a universe of volume  $V = (2\pi)^d e^{d\lambda}$  and energy  $E_n$ . It is the volume that grows and signals a transition to a radiation phase.

correspond to Hagedorn phases of vanishing pressure. During the radiation phases the matter energy jumps to  $E_{\text{max}}$  before falling to the next Hagedorn plateau.<sup>12</sup> In figure 6 one can also see the slight decrease in the amplitude of the oscillations with time.<sup>13</sup>

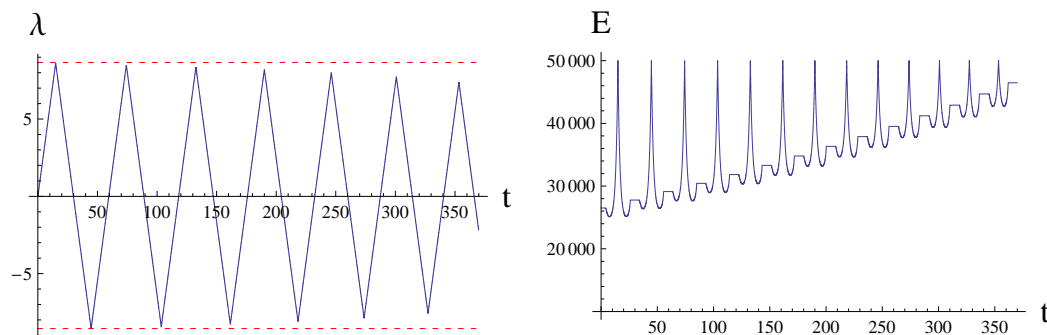


Figure 6: An integration of the equations of motion (6), (8), (31), (32) for  $d = 3$ . As the entropy increases the energy during the Hagedorn phases increases towards  $E_{\text{max}}$  and the size of the oscillations in the scale factor gets smaller (the dashed lines are drawn at constant  $\lambda$ ).

## 6.2 After the Hagedorn era

The dilaton kinetic energy  $E_{\text{max}}$  sets the maximum possible entropy that the system can have and still be in the Hagedorn phase, namely  $S_{\text{max}} = E_{\text{max}}/T_H$ . As entropy is produced during the radiation phases eventually a bounce will occur during which  $S$  exceeds  $S_{\text{max}}$ . At this point a return to a Hagedorn phase is no longer possible.

<sup>12</sup>The small dips in the energy on either side of the plateaus is due to the redshift of energy in an expanding radiation-dominated universe if  $\lambda > 0$ , or the T-dual phenomenon if  $\lambda < 0$ . Eventually either the stretching of winding strings its T-dual takes over and leads to the large spikes in energy. Note that time-reversal invariance is only broken by entropy production during the radiation phases.

<sup>13</sup>In figure 6 we used by hand a slightly larger value for  $T_H$  (larger by a factor of 1.7). This allows us to illustrate the desired effects over a shorter integration time as the phase transitions between Hagedorn and radiation phases occur earlier (smaller  $\lambda$ ) and the interactions are more efficient. The qualitative picture is not altered.

Instead the universe enters a new era which resembles the loitering phase discussed in [13]. The scale factor undergoes oscillations about the minimum of the potential (21), namely<sup>14</sup>

$$V(\lambda) = 2dWe^\lambda + 2dKe^{-\lambda}.$$

Assuming  $\lambda > 0$ , and using the strong-coupling Boltzmann equations (31), (32), the winding strings will gradually annihilate and radiation (momentum modes) will be produced.<sup>15</sup> Eventually all the winding strings will be gone. At that point the oscillations stop and the universe transitions to a radiation-dominated cosmology. With our modified gravity action we may not have the usual radiation-dominated expansion, as one could enter the radiation-dominated era while the scale factors and shifted dilaton are still relativistic ( $\dot{\varphi} \approx -1$ ,  $\dot{\lambda} \approx 1/\sqrt{d}$ ). But eventually the matter energy, or better the combination  $Ee^\varphi$ , becomes small enough that the higher-derivative modifications to the action are unimportant and we go over to a standard radiation-dominated cosmology. The dilaton continues to roll to weak coupling, while the scale factor grows according to

$$e^\varphi \sim \frac{1}{t^{2d/(d+1)}} \quad e^\lambda \sim t^{2/(d+1)}.$$

Somewhat curiously the unshifted dilaton is constant and the scale factor grows just as it would in Einstein gravity.

The whole story can be seen in figure 7 which is simply an extension of figure 6 to later times. It shows the log of the scale factor and the matter energy for a universe evolving through an era of Hagedorn oscillations and a loitering era of potential oscillations before finally entering a radiation-dominated era. In the Hagedorn era the scale factor oscillates about  $\lambda = 0$ , while in the loitering era it oscillates about the minimum in the potential, and in the radiation-dominated era it starts out growing relativistically. The amplitude of the oscillations decreases during the Hagedorn era and increases during the loitering era. The behavior of the matter energy also changes. It has plateaus during the era of Hagedorn oscillations which disappear during the loitering era. (The spikes in the matter energy during the loitering era are simply conversion between kinetic and potential energy.)

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<sup>14</sup>As can be seen from (37), at the moment of Hagedorn exit  $W$  and  $K$  are such that the scale factor sits at the minimum of the potential. For  $\lambda > 0$  the subsequent evolution of  $W$  and  $K$  will tend to shift the minimum to larger radii.

<sup>15</sup>With the weak-coupling Boltzmann equation (30) the strings may never annihilate since the interaction rate turns off as the dilaton rolls to weak coupling [16, 31].

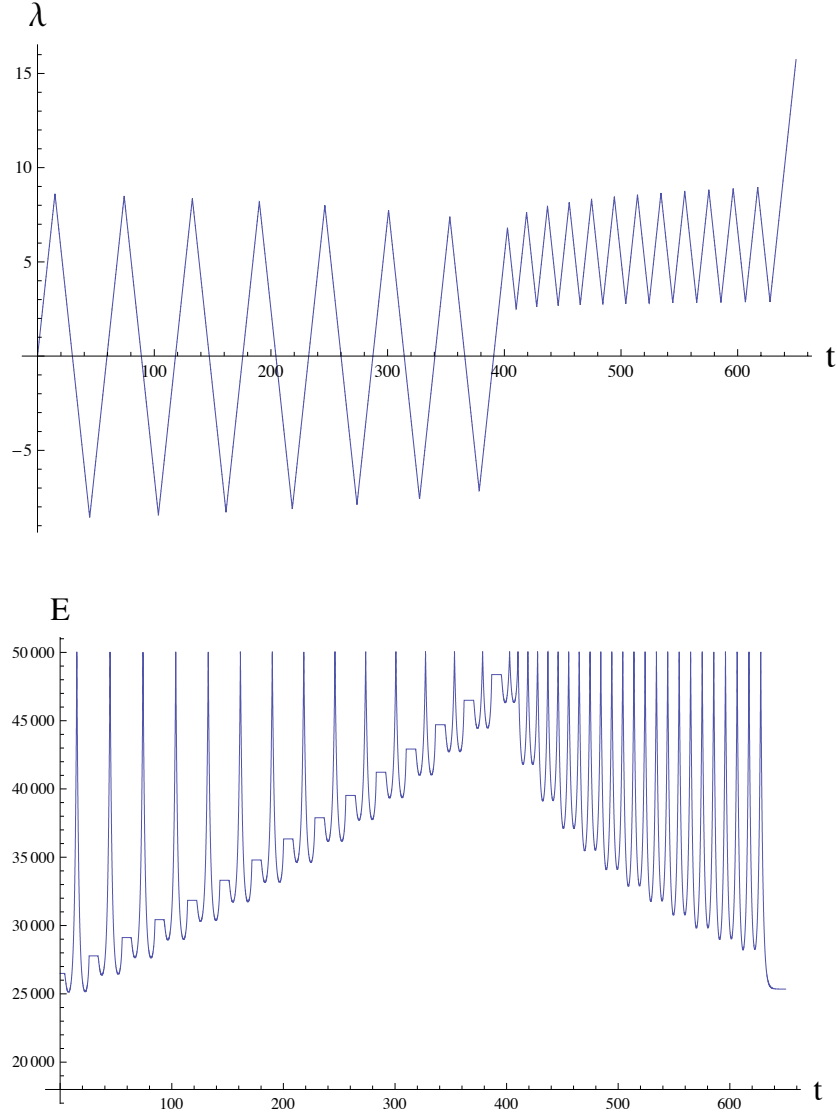


Figure 7: The log of the scale factor and the matter energy in a typical numerical solution. For  $t < 400$  the universe cycles between Hagedorn and radiation phases. For  $400 < t < 640$  the scale factor oscillates about the minimum of its potential while the winding strings gradually annihilate (in practice we use a cutoff value  $W = 1/2$  to specify winding mode annihilation). For  $t > 640$  the universe is radiation-dominated.

## 7 On the BV decompactification mechanism

One might expect the models we have been discussing to provide an ideal setting for realizing the Brandenberger-Vafa mechanism. Indeed this was our original motivation for developing these models. The original BV scenario runs into two difficulties [16, 31]: as the dilaton rolls to weak coupling, the standard Boltzmann equation (30) predicts that string interactions turn off, and one is generically left with a gas of non-interacting strings on a torus of fixed size. Also with the two-derivative effective action (2) the universe has a singularity a finite time in the past, so there is only a limited amount of time for the necessary thermal fluctuations to take place.

Both of these difficulties would seem to be cured in the models we have considered. With the modified Boltzmann equations (31), (32) string interactions do not turn off at late times.<sup>16</sup> Moreover with the modified gravity action (5) the singularity is pushed infinitely far into the past. The oscillating scale factors we have found can be thought of as repeated attempts at decompactifying; if on each bounce there was some probability of decompactifying for  $d \leq 3$ , but vanishing probability for  $d \geq 4$ , then the Brandenberger-Vafa mechanism would work.

This is not, however, the behavior we generically find. Instead in any number of dimensions the era of Hagedorn oscillations eventually ends and the universe transitions to a loitering phase of oscillations about the minimum of the effective potential for  $\lambda$ . Taking  $\lambda > 0$  for purposes of discussion, with the modified Boltzmann equation the winding strings will eventually annihilate and the universe will decompactify. This chain of events can happen for any  $d$ . In this sense the Brandenberger-Vafa mechanism is not operative.

The reason why the BV mechanism seems to be failing is that quantum fluctuations give the winding strings an effective thickness of order  $\sqrt{\alpha'}$  in all spatial dimensions, hence their probability to interact is non-vanishing for any  $d$  and only decreases with  $d$  through a “per volume,”  $\sim e^{-d\lambda}$ , dependence. The original BV argument rested on interactions via (classical) string intersections which did not take into account this (quantum) thickness.

One might still hope that  $d \leq 3$  is favored because the universe might not follow the expected behavior we discussed above. Imagine that due to a thermal fluctuation

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<sup>16</sup>This could also be achieved with the standard Boltzmann equations by introducing a potential to confine the dilaton.

the universe exits a Hagedorn phase with an unusually small number of winding strings. In the subsequent radiation phase perhaps all these strings will annihilate and the universe will decompactify immediately, without additional bounces and without going through a loitering era. Since the annihilation rate (31) falls off rapidly with  $d$ , perhaps this fluctuation-driven mechanism will preferentially decompactify  $d \leq 3$ ?

To address this issue let's study the conditions for decompactifying in a single cycle in the framework we have been using. The probability of decompactifying depends not only on the energy during the Hagedorn phase  $E_n$ , which determines the number of winding strings present at Hagedorn exit, but also on  $E_{\max}$ , which determines how long the subsequent radiation phase will last. For fixed  $E_{\max}$ , smaller values of  $E_n$  – that is, less winding on Hagedorn exit and a larger value of  $\dot{\lambda}$  – will give an increased probability of decompactifying. It's convenient to express this in terms of

$$c = (E_{\max} - E_n)/E_{\max}.$$

Since  $E_n > 0$  we have  $c < 1$ . Taking  $E_{\max} = 10^7$  as an example, we find that 3 dimensions decompactify promptly on Hagedorn exit for  $c \gtrsim 0.984$ . To decompactify 4 dimensions requires  $c \gtrsim 0.9994$ , and to decompactify more dimensions requires slightly larger  $c$ . To translate this into winding numbers on Hagedorn exit we use (35), (37). We find that 3 dimensions decompactify promptly if  $\langle W_n \rangle < 0.507$ , while 4 dimensions decompactify promptly if  $\langle W_n \rangle < 0.504$ . We conclude that, with this value of  $E_{\max}$ , strings are only slightly more efficient at annihilating in  $d = 3$  compared to  $d = 4$ . The only way to decompactify promptly is to exit Hagedorn with essentially no winding (recall that our criterion for no winding was  $W < 0.5$ ).

One could imagine choosing special initial conditions – say a small value of  $E_n$  – to make the winding number small. But this seems against the spirit of the BV mechanism, which should operate starting from generic initial conditions. To quantify just how special the initial conditions have to be, note that the number of Hagedorn-era microstates which decompactify promptly (proportional to the probability of decompactifying) is

$$e^{S_n} = e^{E_n/T_H} \sim e^{-cE_{\max}/T_H}.$$

With  $E_{\max} = 10^7$  the probabilities of having sufficiently small  $E_n$  are tiny, although they do fall off rapidly with  $d$ .

Other types of fluctuations are more likely. For example, even for large  $E_n$  and  $\langle W_n \rangle$ , there might be fluctuations away from the mean that make the winding number vanish. To estimate the probability of this happening, note that for reasonable



distributions of winding numbers the probability of having zero winding on Hagedorn exit scales as

$$\text{Prob.}(\text{no winding}) \sim (1/\langle W_n \rangle)^d.$$

If  $\langle W_n \rangle$  is large then the probabilities are tiny (although again they fall off rapidly with  $d$ ).

There are other interesting types of fluctuations to consider, for example fluctuations in the initial value of  $\lambda$ . For large initial  $\lambda$  strings should be more likely to annihilate in  $d = 3$  than  $d = 4$ , due to the dimension dependence of the cross-section. Although we have not estimated the probability of this happening, it seems unlikely to us that the basic picture will be modified: fluctuations which are large enough to favor  $d = 3$  are also very unlikely to take place. As an alternative approach, one could set initial conditions such that three dimensions decompactify, but this requires careful tuning and violates the spirit of the BV scenario.

We conclude with a few comments on the robustness of our results. We have assumed that the string gas is in thermodynamic equilibrium during the Hagedorn phase. Let's consider the alternative possibility that the winding modes fall out of equilibrium during the Hagedorn phase, well before exit to the radiation phase. To test this we should modify the Boltzmann equations to take into account the fact that string oscillators are highly excited. This was considered in [16] and it amounts to putting a factor of  $E$  in the string cross-sections. With this enhancement, numerical tests for a wide range of energies ( $10^3 - 10^7$ ) showed that during the Hagedorn phase the winding number indeed closely tracked its thermodynamic average. This supports our assumption of a string gas in thermal equilibrium during the Hagedorn phase.<sup>17</sup> Another area of concern is that we have modeled the Hagedorn  $\rightarrow$  radiation transition rather crudely (the average winding number jumps abruptly from  $\sim \sqrt{E}$  to  $\sim E$ ). It should be possible to do a better job with the transition, using results of [25], but we see no reason that an improved treatment of the transition should favor  $d = 3$ .

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<sup>17</sup>Note however that in our models the collective degrees of freedom  $\lambda, \phi$  remain out of equilibrium with the rest of the string gas. Showing that this is realistic, and not say an artifact of our truncation to homogeneous field configurations, deserves further study. We are grateful to Matt Kleban for raising this issue.

## 8 Conclusions

To summarize, we studied the dynamics of a string gas coupled to a modified gravity action. The modified gravity action was set up to avoid singularities, and when coupled to a string gas we found that bouncing and cyclic cosmologies naturally result. Several aspects of our analysis deserve comment and further investigation.

- We postulated a particular form for the modified gravity action (5). It would be interesting to understand to what extent our action captures the effect of  $\alpha'$  corrections in string theory. But we expect that any action which avoids singularities and respects T-duality should lead to qualitatively similar results.
- Modified gravity theories generically have ghosts [32]. A crucial question for future investigation is whether our action for the scale factors (5) can be lifted to a covariant theory, along the lines of [12, 21], and whether the resulting theory is ghost-free [33].
- We described the string gas using modified Boltzmann equations (31), (32) in which we simply dropped the dependence on the (unshifted) dilaton. This could be thought of quasi-phenomenologically, as describing cosmic strings whose interactions do not depend on the dilaton. It could also be thought of as a crude representation of the behavior of either fundamental strings or D-strings [34], given a potential which confines the dilaton to string couplings  $g_s = \mathcal{O}(1)$ .
- Although we developed our models to illustrate some of the features that result from a non-singular string gas cosmology, it would be interesting to study whether they provide a basis for a realistic cosmology. An important step would be to study the spectrum of scalar perturbations resulting from early Hagedorn and loitering eras, extending the work of [35, 36, 37] to the present context.

The models we have discussed provide a remarkably simple realization of bouncing and cyclic cosmologies. With a suitable potential for the dilaton, they also provide a simple realization of the pre-big bang scenario. Let us comment on the two other motivations given in the introduction.

### *Horizon problem*

As we have seen the universe can evolve to a loitering phase in which the scale factor oscillates about the minimum of its potential. If the loitering phase lasts long enough

the entire universe will be in causal contact and might be expected to become quite homogeneous. This would provide a solution to the horizon problem. There are two conditions that must be met.

1. The time-averaged scale factor  $e^{\bar{\lambda}}$  depends on initial conditions while the duration of the loitering phase  $t$  also depends on the string cross-section. The condition for the universe to come in causal contact is  $e^{\bar{\lambda}} \ll t$  which can easily be satisfied by going to weak coupling.
2. Even if the universe is in causal contact we still need to make sure it becomes homogeneous. The condition is that the universe be smaller than the Jeans length,  $e^{\bar{\lambda}} \ll 1/\sqrt{G\rho}$ . Again this can easily be satisfied by going to weak coupling.<sup>18</sup>

Provided these conditions are satisfied any inhomogeneities generated during the Hagedorn phase transitions will be washed out and the universe will eventually approach radiation domination in a state very near thermal equilibrium.<sup>19</sup> In this way our models provide a simple natural resolution of the horizon problem.

#### *Brandenberger-Vafa mechanism*

The Brandenberger-Vafa mechanism is predicated on the idea that winding strings can only annihilate efficiently in  $d \leq 3$  dimensions. Our models evade this reasoning because the universe is expected to enter a loitering phase in which we have strings with a fixed coupling wound on a torus of fixed average size.<sup>20</sup> These wound strings will inevitably annihilate, and the universe will transition to radiation domination, no matter the number of dimensions. In this way the Brandenberger-Vafa mechanism is not operative in the models we have constructed. We leave it as an open challenge to construct a string-inspired model which does preferentially decompactify three dimensions.

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<sup>18</sup>The Jeans length during the Hagedorn phase was studied in [38] and argued to be small. Here we are interested in the Jeans length during the loitering phase for which we adopt the naive estimate  $1/\sqrt{G\rho}$ .

<sup>19</sup>For a study of perturbations in bouncing models see [39].

<sup>20</sup>The fixed coupling is due to our use of the modified Boltzmann equations (31), (32). With the standard Boltzmann equation (30) the strings generically become non-interacting before they can annihilate [16, 31].

## Acknowledgements

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## A Energy conditions in dilaton gravity

Within Einstein gravity a bouncing cosmology requires  $\rho + p < 0$ , a violation of the null energy condition [2, 40]. Here we make the analogous statements for dilaton gravity. A more detailed discussion can be found in [41].

With conventional (two-derivative) dilaton gravity the Friedmann equation and the equations of motion are

$$\frac{1}{2}\dot{\varphi}^2 = \frac{1}{2}d\dot{\lambda}^2 + \frac{1}{8\pi^2}e^\varphi E \quad (39)$$

$$\ddot{\varphi} = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}d\dot{\lambda}^2 + \frac{1}{8\pi^2}e^\varphi P_\varphi \quad (40)$$

$$\ddot{\lambda} = \dot{\varphi}\dot{\lambda} + \frac{1}{8\pi^2}e^\varphi P_\lambda \quad (41)$$

To study a bounce in the scale factor we set  $\dot{\lambda} = 0$ . Then (39) requires  $E \geq 0$ . Equation (40) gives no constraint, while (41) implies that  $P_\lambda$  has the same sign as  $\ddot{\lambda}$ . A string gas can exert pressure of either sign, so it is easy to obtain a bouncing or cyclic scale factor in dilaton gravity coupled to a string gas.

To obtain a bounce in the dilaton is more difficult. Setting  $\dot{\varphi} = 0$  note that (39) requires  $E \leq 0$ . Indeed we had to introduce negative potentials in section 4 to make the dilaton bounce. Equation (41) gives no constraint, while (40) can be rewritten as

$$\ddot{\varphi} = -\frac{1}{8\pi^2}e^\varphi(E - P_\varphi). \quad (42)$$

The sign of  $\ddot{\varphi}$  is correlated with the sign of  $E - P_\varphi$ . In particular  $\ddot{\varphi} > 0$  requires  $E - P_\varphi < 0$ , the dilaton gravity analog of violating the null energy condition.<sup>21</sup>

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<sup>21</sup>Due to the wrong-sign kinetic term for the dilaton we inserted a minus sign in our definition of  $P_\varphi$  below (6).

One can likewise study the conditions for a bounce in the Einstein-frame scale factor  $\lambda_E = -(\varphi + \lambda)/(d - 1)$ . When  $\lambda_E$  bounces we have  $\dot{\varphi} = -\dot{\lambda}$  and (39) requires  $E < 0$ . Adding (40) and (41) gives

$$\ddot{\lambda}_E = \frac{d-1}{8\pi^2} e^\varphi (E - P_\varphi - P_\lambda).$$

Thus the sign of  $\ddot{\lambda}_E$  is correlated with the sign of  $E - P_\varphi - P_\lambda$ .

The use of our higher-derivative modified action for dilaton gravity does not significantly change these string-frame results. In fact the only change is that (42) is replaced with

$$\gamma_\lambda \ddot{\varphi} = -\frac{1}{8\pi^2} e^\varphi (E - P_\varphi \gamma_\lambda)$$

so the sign of  $\ddot{\varphi}$  at a bounce is correlated with the sign of  $E - P_\varphi \gamma_\lambda$ .

## B Exiting Hagedorn with a two-derivative action

In this appendix we study the space of initial conditions which allows the universe to start in an initial Hagedorn phase and subsequently exit. The discussion is based on the two-derivative effective action  $S_0$  given in (2).

### B.1 Requirements on initial conditions in the Hagedorn phase

The starting point for the universe in string gas models is a high temperature equilibrium phase near the self-dual radius. In this initial Hagedorn phase, to a good approximation, the pressure vanishes and the energy is constant. The assumption of equilibrium is quite reasonable, as the interaction rates of strings are enhanced due to their large oscillator numbers (lots of string in a small space). As stated, the universe is taken to be near the self-dual radius,  $\lambda \approx 0$ , and nearly static,  $\dot{\lambda} \approx 0$ .

Now let's consider fluctuations about this equilibrium configuration. Suppose a thermal fluctuation allows  $d$  dimensions to grow, and for simplicity take this part of

the universe to be isotropic. The equations of motion become ( $P_\lambda = P_\varphi = 0$ )

$$\begin{aligned} E &= (2\pi)^2 e^{-\varphi} (\dot{\varphi}^2 - d\dot{\lambda}^2) \\ \ddot{\varphi} &= \frac{1}{2} (\dot{\varphi}^2 + d\dot{\lambda}^2) \\ \ddot{\lambda} &= \dot{\varphi}\dot{\lambda} \end{aligned}$$

With  $E$  constant, these can be solved exactly, with

$$\begin{aligned} \varphi(t) &= \log \left[ \frac{e^{\varphi_0}}{(E/16\pi^2)e^{\varphi_0}t^2 - \dot{\varphi}_0 t + 1} \right] \\ \lambda(t) &= A + \frac{1}{\sqrt{d}} \log \left[ \frac{(E/8\pi^2)e^{\varphi_0}t - (\dot{\varphi}_0 + \sqrt{d}\dot{\lambda}_0)}{(E/8\pi^2)e^{\varphi_0}t - (\dot{\varphi}_0 - \sqrt{d}\dot{\lambda}_0)} \right] \end{aligned}$$

The subscripts 0 denote values at  $t = 0$  and are assumed to satisfy the Hamiltonian constraint. In these solutions, the constant  $A$  is the asymptotic value of the scale factor, related to initial conditions by

$$A = \lambda(t \rightarrow \infty) = \lambda_0 + \frac{1}{\sqrt{d}} \log \left[ \frac{-(\dot{\varphi}_0 - \sqrt{d}\dot{\lambda}_0)}{-(\dot{\varphi}_0 + \sqrt{d}\dot{\lambda}_0)} \right]. \quad (43)$$

(We are considering  $\dot{\varphi}_0 < 0$  and  $\dot{\lambda}_0 > 0$ , while for positive matter energy we must have  $\sqrt{d}\dot{\lambda}_0 < |\dot{\varphi}_0|$ , so the arguments of the logarithms are positive.)

At some radius larger than  $\lambda_0$  the universe is expected to fall out of equilibrium and enter a new “large radius” phase. An exit condition, since the energy in the Hagedorn phase is constant, can be expressed in the form

$$\rho(\lambda_{\text{exit}}) = \rho_H \quad (44)$$

where

$$\rho(\lambda) = \frac{E e^{-d\lambda}}{(2\pi)^d}$$

is the energy density in the universe and  $\rho_H$  is a characteristic energy density of order  $\alpha'^{-(\frac{d+1}{2})}$ . We will be more precise about the value of  $\rho_H$  below. With the Hagedorn phase solution above, in order to exit to a radiation era we need

$$\lambda(t \rightarrow \infty) > \lambda_{\text{exit}}.$$

Using (43) and (44) this can be written conveniently in terms of the variable  $x \equiv \frac{\sqrt{d}\dot{\lambda}_0}{|\dot{\varphi}_0|}$  as  $x > (r-1)/(r+1)$  where  $r \equiv (\rho(\lambda_0)/\rho_H)^{1/\sqrt{d}}$  is the ratio of the initial energy density to the critical Hagedorn density. To be in the Hagedorn era at  $t=0$  we need  $r > 1$ . Also we need  $x < 1$  for positive matter energy.

## B.2 Testing the phase space of initial conditions

The equations of motion have four initial conditions, which can be chosen to be the initial volume  $V_0 = (2\pi)^d e^{d\lambda_0}$ ,  $\varphi_0$ ,  $\dot{\varphi}_0$  and  $x$  (instead of  $\dot{\lambda}$ , once  $\dot{\varphi}_0$  is fixed). The variable  $x$  reflects the initial “boost” of the scale factors. Fixing the first three initial conditions, we then ask whether a given  $x$  can drive the system out of the Hagedorn phase.

The conditions on initial conditions such that the system starts in the Hagedorn era ( $\rho(\lambda_0)/\rho_H > 1$ ) and exits to the large radius era ( $x > (r-1)/(r+1)$ ) can be written as

$$\begin{aligned} f_1 &\equiv K_0(1-x^2) - 1 > 0 \\ f_2 &\equiv x - \frac{K_0^{1/\sqrt{d}}(1-x^2)^{1/\sqrt{d}} - 1}{K_0^{1/\sqrt{d}}(1-x^2)^{1/\sqrt{d}} + 1} > 0 \end{aligned}$$

where

$$K_0(\varphi_0, \dot{\varphi}_0, V_0) \equiv \frac{(2\pi)^2 e^{-\varphi_0} \dot{\varphi}_0^2}{\rho_H V_0}.$$

In the Hagedorn phase the entropy in matter is to a good approximation proportional to the energy. We use this in all that follows. With  $S = E/T_H$ , the distribution of  $x$  once the other three initial conditions are fixed is a Gaussian,

$$d(x) \sim e^{-(4\pi^2 e^{-\varphi_0} \dot{\varphi}_0^2 / T_H) x^2}. \quad (45)$$

Now let’s be more precise about the condition for Hagedorn exit. In the Hagedorn phase the string gas is taken to be in equilibrium at temperature  $T_H$ . This is similar to having a black hole in thermal equilibrium with the surrounding radiation, something which can only happen in finite volume. The constraints on the volume in which such equilibrium can be maintained have been studied, and for a string gas in  $d$  spatial dimensions read [42, 43, 44]

$$V_d < \frac{E}{c_d T_c^{d+1}}. \quad (46)$$

Here  $T_c$  is the temperature and

$$c_d = 128 \frac{2d! \zeta(d+1)}{(4\pi)^{d/2} \Gamma(d/2)} (2 - 2^{-d}) \quad (47)$$

is the Stefan-Boltzmann constant for 128 fermionic and 128 bosonic massless degrees of freedom. This suggests that

$$\rho_H \simeq c_d T_H^{d+1} \equiv \rho_c(d). \quad (48)$$

Below this energy density the string gas will decay to radiation.

To study the probability of exiting Hagedorn we fix the initial volume to be  $V_0 = (2\pi)^d$ , that is, we study fluctuations when the universe is at the self dual radius, and survey the remaining 3-dimensional space of initial conditions. Figure 8 shows the overlap of the region where  $f_1, f_2 > 0$  with the region where  $x$  is taken to be within 3 standard deviations of its mean in the distribution (45). The space of “good” initial conditions is obviously restricted. Figure 9 is similar but with  $\rho_H = 10\rho_c(d)$  and  $\rho_H = 0.01\rho_c(d)$ . The fact that the plots are similar shows that our results are not sensitive to our estimate for  $\rho_H$ . All the plots are for  $d = 3$ , but they change only slightly for different  $d$ .

One may still question whether there is indeed a problem with initial conditions. A fluctuation to a large radius era, where winding modes will want to annihilate and radiation will take over, could be rare but if the system is kept in the Hagedorn phase such a fluctuation will eventually occur. But this does not happen if one takes the effective dilaton gravity action  $S_0$  seriously: it implies that the universe begins from an initial singularity and has only a finite amount of time before the dilaton rolls to weak coupling.

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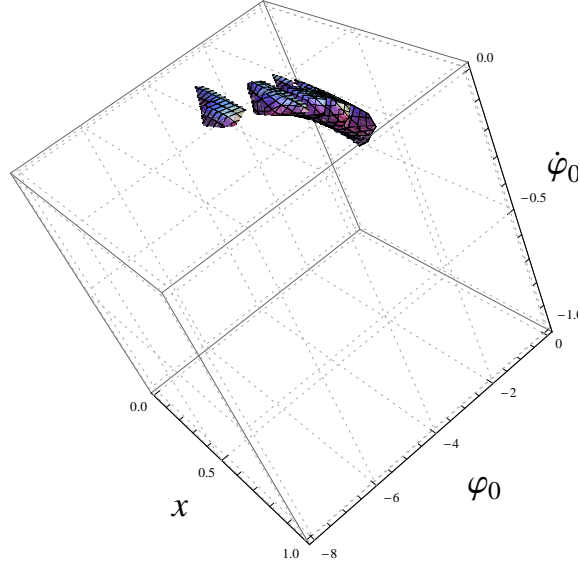


Figure 8: The shaded region is the space of initial conditions for which exit from the Hagedorn phase is possible. It is the region where  $f_1$  and  $f_2$  are positive and  $x$  is within 3 standard deviations of its mean. We set  $\rho_H = \rho_c(d)$  and  $d = 3$ .

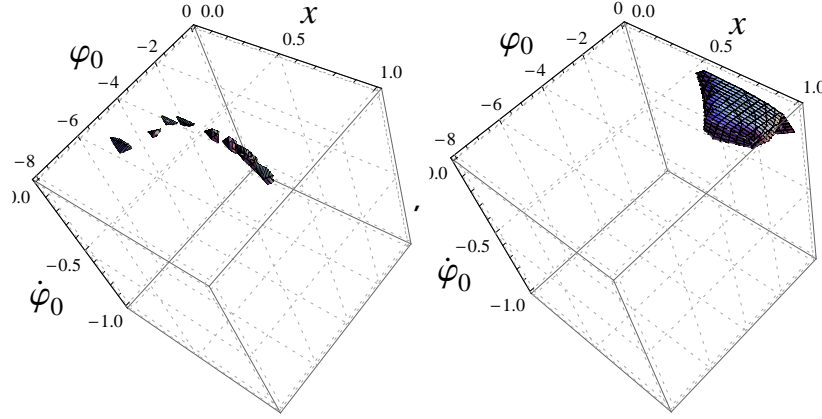


Figure 9: Same as figure 8 but with  $\rho_H = 10\rho_c(d)$  (left) and  $\rho_H = 0.01\rho_c(d)$  (right).

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